You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions. All numbers appearing in the test are complex numbers and all functions are from $\mathbb{C}$ to $\mathbb{C}$.
Exercises 1-3 are on residue computaions, 4-6 are on Laurent expansions and 7-9 are general. Choose and solve one exercise in each group.

Name: ______________________________

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	12	10	10	10	10	8	8	12	90
Score:										

1. (10 points) Use complex analysis to evaluate

$$I_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(n\theta)}{\sin(\theta)}d\theta \tag{1}$$

for every positive n.

Solution: If $z = e^{i\theta}$ we have that:

$$\sin(n\theta) = \frac{z^n - z^{-n}}{2i} \tag{2}$$

so that we can write:

$$I_n = \frac{1}{2i\pi}\int_\gamma \frac{z^n - z^{-n}}{z(z - z^{-1})}dz = \frac{1}{2i\pi}\int_\gamma \frac{z^{2n}-1}{z^n(z^2-1)}dz \tag{3}$$

where $\gamma = \{e^{i\theta} \,|\, 0 \le \theta \le 2\pi\}$. Observe that

$$\frac{z^{2n}-1}{z^2-1} = \sum_{k=0}^{n-1} z^{2k} \tag{4}$$

Only the term with $2k = n - 1$ in the previous sum gives a non zero contribution when inserted in the integral. Thus we have:

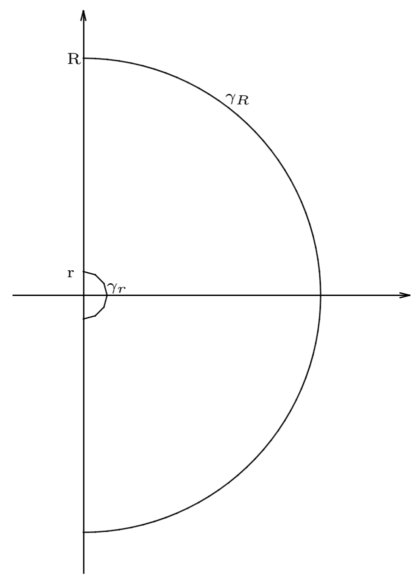
$$I_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \tag{5}$$

2. (12 points) Compute

$$\int_0^\infty \frac{(\log(x))^2}{2+x^2} dx. \tag{6}$$

(*Hint:* Write the integral as an integral on a path containing the complex axis.)

Solution: we choose for $\log(z)$ the standard branch. Let γ be the curve:



We have

$$\int_\gamma \frac{(\log(z))^2}{2-z^2} dz = \frac{2i\pi}{2\sqrt{2}}(\log\sqrt{2})^2 \tag{7}$$

since there is only one pole in the curve at $z=\sqrt{2}$. On the other hand we have

$$\begin{aligned} \int_\gamma \frac{(\log(z))^2}{2-z^2} dz &= i\int_r^R \frac{(\log(x)+i\frac{\pi}{2})^2}{2+x^2} dx + i\int_{-R}^{-r} \frac{(log(|x|)-i\frac{\pi}{2})^2}{2+x^2} dx \\ &+ \int_{\gamma_R} \frac{(\log(z))^2}{2-z^2} dz + \int_{\gamma_r} \frac{(\log(z))^2}{2-z^2} dz \end{aligned} \tag{8}$$

Reasoning exactly like in the example in the book we get that the limit for $R\to\infty$ and $r\to 0$ of the last two integrals is 0. Thus we have, after taking the limits, that

$$\frac{2\pi}{2\sqrt{2}}(\log\sqrt{2})^2 = 2\int_0^\infty \frac{(\log(x))^2}{2+x^2} dx - \frac{\pi^2}{2}\int_0^\infty \frac{1}{2+x^2} dx \tag{9}$$

Finally we get

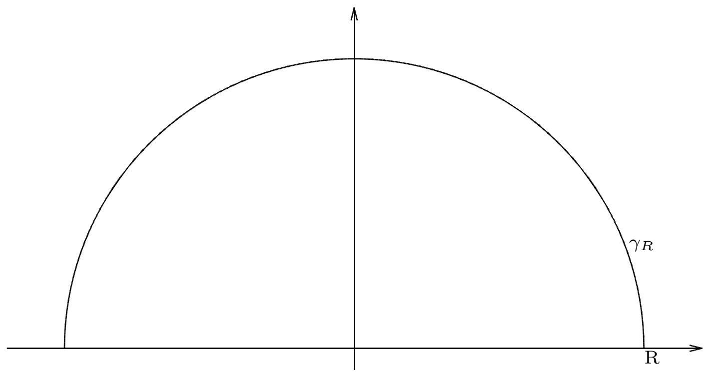
$$\int_0^\infty \frac{(\log(x))^2}{2+x^2} dx = \frac{\pi\sqrt{2}}{16}((\log 2)^2+\pi^2) \tag{10}$$

3. (10 points) Compute the integral

$$\int_0^\infty \frac{x\sin(ax)}{(x^2+1)^2}dx \tag{11}$$

where $a > 0$.

Solution: Let γ be the curve:



We have

$$\int_\gamma \frac{ze^{iaz}}{(z^2+1)^2}dz = \int_{-R}^{R} \frac{xe^{iax}}{(x^2+1)^2}dx + \int_{\gamma_R} \frac{ze^{iaz}}{(z^2+1)^2}dz \tag{12}$$

where γ_R is the semicircle. Observe that the limit for $R \to \infty$ of the integral on γ_R is 0. On the other hand we have

$$\int_\gamma \frac{ze^{iaz}}{(z^2+1)^2}dz = \frac{i\pi}{2}ae^{-a}. \tag{13}$$

We thus have

$$\int_0^\infty \frac{x\sin(ax)}{(x^2+1)^2}dx = \frac{\pi}{4}ae^{-a} \tag{14}$$

4. (10 points) Show that the function

$$f(z) = \frac{\cos(z)}{z^2} \tag{15}$$

is the derivative of a function F analytic in $\mathbb{C}\backslash\{0\}$. Write the Laurent series for F around $z = 0$.

Solution: Since every closed path in $\mathbb{C}\backslash\{0\}$ is homotopically equivalent to $\gamma_n = \{e^{in\theta} \,|\, 0 \leq \theta \leq 2\pi\}$ for some n, it is enough to observe that, due to simmetry,

$$\int_{\gamma_1} \frac{\cos(z)}{z^2} dz = i \int_0^{2\pi} \cos\left(e^{i\theta}\right) e^{-i\theta} d\theta = 0 \tag{16}$$

to obtain that the primitive F exists and is analytic on $\mathbb{C}\backslash\{0\}$. We clearly have

$$f(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} z^{2(n-1)} \tag{17}$$

so that we get that F must be, apart from an additive constant,

$$F(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n)!} z^{2n-1} \tag{18}$$

5. (10 points) Consider the function f given by the power series

$$f(z) = \sum_{k=0}^{\infty} (k^2 + 5k - 1) z^k \tag{19}$$

Show that f can be extended to a meromorphic function on $\mathbb{C}$. Find the Laurent expamsion of f around $z = 1$.

Solution: Obeserve that

$$k^2 + 5k - 1 = (k+2)(k+1) + 2(k+1) - 5 \tag{20}$$

so that

$$f(z) = \sum_{k=0}^{\infty} (k+2)(k+1) z^k + 2 \sum_{k=0}^{\infty} (k+1) z^k - 5 \sum_{k=0}^{\infty} z^k = \frac{1}{(1-z)^3} - \frac{2}{(1-z)^2} - \frac{5}{1-z} \tag{21}$$

6. (10 points) Consider the function

$$f(z) = \frac{\exp\frac{1}{z}}{(2-z)^2} \tag{22}$$

Find the positive part of the Laurent expansion of f around $z = 0$.

Solution: Observe that, apart $z = 0$, f has a unique pole of order 2 in $z = 2$. We can thus write:

$$f(z) = \frac{\exp\frac{1}{z} - \sqrt{e} + \sqrt{e}(z-2)/4}{(z-2)^2} + \sqrt{e}\frac{1}{(z-2)^2} - \frac{\sqrt{e}}{4}\frac{1}{z-2} \tag{23}$$

The function

$$g(z) = \frac{\exp\frac{1}{z} - \sqrt{e} + \sqrt{e}(z-2)/4}{(2-z)^2} \tag{24}$$

is analytic in $\mathbb{C}\backslash\{0\}$ and $\lim_{z\to\infty}|f(z)| = 0$. This implies that its Laurent expansion contains only term with negative powers of z. It follows that the positive part of the Laurent expansion is obtained expanding

$$h(z) = \sqrt{e}\frac{1}{(z-2)^2} + \frac{\sqrt{e}}{4}\frac{1}{z-2} = \frac{\sqrt{e}}{4}\sum_{k=0}^{\infty}\frac{k+2}{2^{(k+1)}}z^k \tag{25}$$

7. (8 points) Let f be a non constant meromorphic function on $\mathbb{C}$ such that $f(z) \neq 0$ for all z. Prove that there exists a sequence z_n such that $\lim_{n\to\infty} f(z_n) = 0$.

Solution: From the hypothesis it follows that $g(z) = 1/f(z)$ is an entire function since it has no pole and it has only removable singularities where $f(z)$ has poles. It follows that g cannot be bounded since f, and thus g, is not constant. This means that there exists a sequnce z_n such that $\lim_{n\to\infty} |g(z_n)| = \infty$ which implies that $\lim_{n\to\infty} f(z_n) = 0$.

8. Let f be a meromorphic function.

   (a) (8 points) Suppose that f is analytic in $A_R = \{z \,|\, |z| > R\}$ for some $R > 0$. Show that f can be written as the ratio of two entire functions.

Solution: Observe first that the complement of A_R is the closed disk D_R of radius R centered at the origin. Since D_R is compact f can have at most a finite number of poles. Let p_k, $k = 1, \ldots, m$, be the poles and n_k their molteplicity. We know that we can write f as:

$$f(z) = \sum_{k=1}^{m} \frac{q_k(z)}{(z - p_k)^{n_k}} + r(z) \tag{26}$$

where q_k is a polynimial of degree less than n_k and r is an entire function. It is now enough to collect the denominators to obtain

$$f(z) = \frac{r(z) + \sum_k q_k(z) \prod_{i \neq k} (z - p_i)^{n_i}}{\prod_k (z - p_k)^{n_k}} \tag{27}$$

Clearly both the denominator and the numerator are entire function.

   (b) (12 points (bonus)) Is the conclusion still valid if we remove the assumption that f is analytic in D_R?

Solution:

9. (12 points) Let f be an analytic function in $D = \{z \,|\, |z| < 1\}$ such that $f(z) \neq 0$ for all $z \in D$. Show that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-it}\log\left|f\left(re^{it}\right)\right| dt = \frac{rf'(0)}{2f(0)} \tag{28}$$

(*Hint:* Obeserve that $f'/f = (\log f)'$ and $\log|f| = \Re(\log f) = (\log f + \overline{\log f})/2$.)

Solution: Since f is never 0 we have that $g(z) = \log f(z)$ is well defined and analityc for $z \in D$. Let $\gamma = \{re^{i\theta} \,|\, 0 \leq t \leq 2\pi\}$, we have

$$0 = \int_\gamma g(z)dz = ir\int_0^{2\pi} \log\left(f\left(re^{it}\right)\right)e^{it}dt \tag{29}$$

Taking the complex conjugate we get

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{\log\left(f\left(re^{it}\right)\right)}e^{-it}dt = 0 \tag{30}$$

On the other hand

$$\frac{f'(0)}{f(0)} = \frac{1}{2i\pi}\int_\gamma \frac{g(z)}{z^2}dz = \frac{1}{2\pi r}\int_0^{2\pi} \log\left(f\left(re^{it}\right)\right)e^{-it}dt \tag{31}$$

Summing this two equations give the thesis.