You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions. All numbers appearing in the test are complex numbers and all functions are from $\mathbb{C}$ to $\mathbb{C}$.

When returning your Exam, you must return also this page, signed. Thanks.

To solve the Exam problems, I have not collaborated with anyone or used any source except class notes and the textbook.

Name: ______________________________

Question:	1	2	3	4	5	6	Total
Points:	25	20	25	20	40	20	150
Score:							

1. (25 points) Let f be a non constant analytic function defined on the unit disk D. Assume that $\mathrm{Re} f(z) > 0$ for every $z \in D$ and $f(0) = 1$. Prove that

$$|f(z)| \leq \frac{1+|z|}{1-|z|}$$

What can you say if there exists $z_0 \neq 0$ in D such that

$$|f(z_0)| = \frac{1+|z_0|}{1-|z_0|}$$

(**Hint:** remember that $|w+1| \geq |w|$ if $\mathrm{Re}\, w > 0$ and $|w+v| \geq \big||w|-|v|\big|$. Use Schwartz Lemma.)

2. (20 points) Let $f(z)$ be a meromorphic function such that

$$|f(z)| \leq \left(\frac{2|z|}{|z-1|}\right)^{\frac{3}{2}}.$$

Show that $f \equiv 0$.

3. (25 points) Let $G \subset \mathbb{C}$ be an open region and let $\{f_n\}_{n\geq 0}$ be a sequnce of analytic functions. Show that the sequence f_n converge uniformly on every compact subset of G if and only if

$$\left\{\int_\gamma f_n(z)dz\right\}_{n\geq 0}$$

is a Cauchy sequence for every piecewise smooth curve γ.

4. (20 points) Let $a_i(t)$, $i = 0, \dots n$, be continuous functions of t. Consider the family of polynomials

$$p(z;t) = \sum_{i=0}^{n} a_i(t) z^i$$

and assume that $p(z;0)$ has exactly k zeros in the disk $|z - c| < R$ and no zero on the circle $|z - c| = R$. Prove that for t small enough $p(z;t)$ has exactly k zeros, counted with molteplicity, inside the disk $|z - c| < R$. (**Hint:** you can use Rouché Theorem.)

5. Compute the following integrals:
   (a) (20 points)
$$\int_0^\infty \frac{\cos x}{a^2 + x^2} dx.$$

(b) (20 points)

$$\int_0^\infty \frac{(\log x)^3}{1+x^2}dx.$$

6. (20 points) Let $\hat{f}(k)$, $k \in \mathbb{R}$, be a real and continuous function such that

$$|\hat{f}(k)| \leq Ce^{-\rho|k|}$$

Show that

$$f(z) = \int_{-\infty}^{\infty} e^{ikz} \hat{f}(k) dk$$

in analytic in the strip $\{\operatorname{Im} z < \rho\}$.