You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions. All numbers appearing in the test are complex numbers and all functions are from $\mathbb{C}$ to $\mathbb{C}$.

Name: ______________________________

Question:	1	2	3	4	5	Total
Points:	10	10	10	40	10	80
Score:						

1. (10 points) Find the set $D \in \mathbb{C}$ where the following series converges:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n^2} \exp\left(\frac{nz}{z-i}\right).$$

Solution: Since

$$\exp\left(\frac{nz}{z-i}\right) = \exp\left(\frac{z}{z-i}\right)^n$$

we need

$$\left|\exp\left(\frac{z}{z-i}\right)\right| < 1 \qquad \text{or} \qquad \mathrm{Re}\left(\frac{z}{z-i}\right) < 0$$

Let $g(z) = z/(z-i)$ and observe that $f(0) = 0$, $f(\infty) = 1$ and $f(i) = \infty$ so that g maps the immaginary line into the real line and $\mathrm{Im}(z) > 0$ into $\mathrm{Re}(z) < 0$. Thus the series defining $f(z)$ converges uniformly in $\mathrm{Im}(z) > 0$.

2. (10 points) Let $D = \{z \,|\, |z| < 1\}$. Is there an analityc function f from D to $\mathbb{C}$ such that for every $n \in \mathbb{N}$

   a)
   $$f\left(\frac{1}{n}\right) = \frac{1}{n^2} \qquad \text{and} \qquad f\left(-\frac{1}{n}\right) = \frac{1}{n^2}$$

   b)
   $$f\left(\frac{1}{n}\right) = \frac{1}{n^3} \qquad \text{and} \qquad f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$$

   Justify your answer clearly.

   Solution: Point a) is trivial: it is enough to take $f(z) = z^2$. Regarding point b) from the value on $1/n$ it follows that $f(z) = z^3$ so that it cannot satisfy the condition on $-1/n$.

3. (10 points) Let f be an entire function such that

$$|f(z)| < 2|z|^{\frac{3}{2}} - 1 \tag{1}$$

for $|z| > 1$. Give a characterization of f as complete as you can.

Solution: Write $f(z) = a_0 + a_1 z + z^2 g(z)$ with $g(z)$ entire so that $g(z) < M$, for a suitable M, on $|z| < 1$. Moreover

$$|g(z)| = \left|\frac{f(z) - a_0 - a_1 z}{z^2}\right| < \frac{2}{\sqrt{|z|}} + \frac{1 + a_0}{z^2} + \frac{a_1}{z}$$

for $|z| > 2$. Thus $g(z)$ is bounded so that $g(z) = c$ and, since $\lim_{z\to\infty} g(z) = 0$, we have $g \equiv 0$. Thus

$$f(z) = a_0 + a_1 z$$

Observe that

$$\max_{|z|=r} |a_0 + a_1 z| = |a_0| + |a_1| r$$

so that it must be $|a_0| + |a_1| r < 2r^{\frac{3}{2}} - 1$ for $r > 1$. For $r = 1$ this tells that $|a_0| + |a_1| < 1$. Moreover for $r = 1$, $(|a_0| + |a_1| r)' < (2r^{\frac{3}{2}} - 1)'$ if $|a_1| < 1$. Since $2r^{\frac{3}{2}} - 1$ is a convex function we have that it is enough to ask

$$|a_0| + |a_1| < 1.$$

4. Let
$$f(z) = \int_{-1}^{1} \frac{e^{-x^2}}{x-z}dx.$$
and let $I = (-1, 1)$ considered as a subset of $\mathbb{C}$.

(a) (10 points) Prove that f is analytic on $\mathbb{C}\backslash\bar{I}$.

Solution: If $z \notin \bar{I}$ then
$$\phi(x, z) = \frac{e^{-x^2}}{x-z}$$
is continuous for $x \in \bar{I}$ and $z \in \mathbb{C}$ so that the statement follows immediately from Proposition 2.1 in the book.

(b) (10 points) Let $z_n = u + iv_n$ with $u, v \in \mathbb{R}$, $|u| < 1$ and $v_n > 0$ such that $v_n \to 0$ as $n \to \infty$. Observe that this implies that $z_n \to z \in I$ from "above". Prove that $\lim_{n\to\infty} f(z_n)$ exists. (**Hint:** write $f(z)$ as a line integral on a curve γ with trace I. Complete the curve with a new curve σ such that $\gamma + \sigma$ is a closed curve. Finally use Chauchy theorem to write f as a line integral over σ plus ...)

Solution: Let σ be the curve $\{e^{\pi it} \,|\, 0 < t < 1\}$ and $\gamma = \{2t - 1 \,|\, 0 < t < 1\}$. We have, for $\mathrm{Im}(z) > 0$ and $|z| < 1$,
$$f(z) = \int_{\gamma} \frac{e^{-w^2}}{w-z}dw \qquad \int_{\gamma+\sigma} \frac{e^{-w^2}}{w-z}dw = 2\pi i e^{-z^2}$$
so that
$$f(z) = \int_{-\sigma} \frac{e^{-w^2}}{w-z}dw + 2\pi i e^{-z^2}$$
The above expression is clearly continuous so that
$$\lim_{n\to\infty} f(z_n) = \int_{-\sigma} \frac{e^{-w^2}}{w-u}dw + 2\pi i e^{-u^2}$$

(c) (10 points) Let $D = \{z \,|\, |\mathrm{Re}(z)| < 1\}$ and $D^+ = \{z \in D \,|\, \mathrm{Im}(z) > 0\}$, $D^- = \{z \in D \,|\, \mathrm{Im}(z) < 0\}$. Show that there exists a function f_+ analytic in D such that $f_+(z) = f(z)$ for $z \in D^+$. Analogously show that there exists f_- analytic in D such that $f_-(z) = f(z)$ for $z \in D^-$.(**Hint**: use the expression you found on the previous point to define f_+ on D^- and ...)

Solution: With the notation of the previous point let $C = \{z \in D^+ \,|\, |z| < 1\}$. Then

$$g(z) = \int_{-\sigma} \frac{e^{-w^2}}{w-z} dw + 2\pi i e^{-z^2}$$

is analytic on $D^- \cup I \cup C$. Clearly $f \equiv g$ on C so that we can define $f_+(z) = f(z)$ for $z \in D^+ \cup I$ and $f_+(z) = g(z)$ for $z \in D^-$.

The construction of f_- is completely analogous using

$$g'(z) = \int_{-\sigma'} \frac{e^{-w^2}}{w-z} dw - 2\pi i e^{-z^2}$$

where $\sigma' = \{e^{-\pi i t} \,|\, 0 < t < 1\}$.

(d) (10 points) Is it true that $f_+(z) = f_-(z)$ for $z \in I$? If not, compute the difference.

Solution: For $z \in I$ we have

$$f_+(z) - f_-(z) = g(z) - g'(z) = \int_{\sigma' - \sigma} \frac{e^{-w^2}}{w-z} dw + 4\pi i e^{-z^2} = 2\pi i e^{-z^2}$$

5. (10 points) For $0 < \theta < \pi$, compute

$$\frac{1}{2\pi i}\int_\gamma \frac{z^n}{1 - 2z\cos(\theta) + z^2}dz$$

where $\gamma = \{e^{it} \,|\, 0 < t < 2\pi\}$.

Solution: Observe that $1 - 2z\cos(\theta) + z^2 = 0$ for $z = e^{\pm i\theta}$ so that

$$\frac{1}{2\pi i}\int_\gamma \frac{z^n}{1 - 2z\cos(\theta) + z^2}dz = \frac{e^{in\theta}}{e^{-i\theta} - e^{i\theta}} + \frac{e^{-in\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(n\theta)}{\sin(\theta)}$$