You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions. All numbers appearing in the test are complex numbers and all functions are from $\mathbb{C}$ to $\mathbb{C}$.

Name: ______________________________

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	5	10	10	12	15	15	10	15	10	10	122
Score:												

1. (10 points) Let f_n be a sequence of analytic functions on $\Omega \subset \mathbb{C}$. Assume that f_n converge to f uniformly on every compact subset of Ω and that f is not identically 0. Prove that if f has a simple 0 at $a \in \Omega$ then there exists N such that for every $n > N$ there exists a_n with $f_n(a_n) = 0$. Moreover $\lim_{n\to\infty} a_n = a$.

Solution: Let r be small enough so that the only 0 of f in $B(a, 2r) \subset \Omega$ is a and let $\delta = \inf_{z\in\partial B(a,r)} |f(z)|$. For n large enough we have $|f_n(z) - f(z)| \leq \delta/2$ so that f_n is never 0 on $\partial B(a, r)$. Clearly f_n' converge uniformely to f' on Ω. We thus have that:

$$\frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f_n'(z)}{f_n(z)} dz \longrightarrow_{n\to\infty} \frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f_n(z)}{f(z)} dz = 1 \tag{1}$$

Since the above integral is always an integer this means that for n large enough:

$$\frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f_n'(z)}{f_n(z)} dz = 1 \tag{2}$$

This proves the first part of the thesis. Observe now that we can repeat this argument for every r small enough so that, for every ϵ, a_n is definitevely in $B(a, \epsilon)$.

2. (5 points) Let $f(z)$ be an analytic function on $E = \{z \,|\, 0.5 < |z| < 2\}$. Define $F(x) = f(\exp(ix))$. Compute the Fourier series of F. What can you say on the Fourier coefficient?

Solution: For $|z| = 1$ we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{3}$$

Calling $z = e^{ix}$ we get

$$F(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx} \tag{4}$$

that is the Fourier series for F. Since f is analytic in E we can say that $a_n \leq C2^{-|n|}$ for some contant C.

3. (10 points) Let

$$f(z) = \prod_{n=1}^{\infty} \cos\left(\frac{z}{n}\right) \tag{5}$$

Prove that f is entire. Find the z such that $f(z) = 0$. (**Hint:** you just need an estimate of $\cos(z) - 1$ for $|z|$ small.)

Solution: To prove the thesis we need to show that:

$$\sum_{n=1}^{\infty} \left[\cos\left(\frac{z}{n}\right) - 1\right] \tag{6}$$

converges absolutely and uniformely on every disk $\{z \,|\, |z| \leq R\}$. Observe that, if $|z| < 1/2$, we have

$$|1 - \cos(z)| \leq \sum_{n=1}^{\infty} \frac{|z^{2n}|}{(2n)!} \leq |z|^2 \sum_{n=0}^{\infty} |z|^{2n} \leq 2|z|^2 \tag{7}$$

Thus, given R, if $n > 2R$ and $|z| < R$ we have that $1 - \cos\left(\frac{z}{n}\right) \leq \frac{2R^2}{n^2}$. By the Weierstrass M-test the series converges uniformly and absolutely.

The only zeroes of f are the z such that $\cos\left(\frac{z}{n}\right) = 0$ for some n This means $z = n(2k+1)\pi/2$ *i.e.* all integer multiple of $\pi/2$.

4. (10 points) Let

$$f(z) = \log\left(\frac{z}{z-1}\right) \tag{8}$$

Prove that f is analytic in $R = \{z \mid |z| > 1\}$. Find the Laurent expansion of f around $z = 0$

Solution: The map $h(z) = z/(z-1)$ is a Möbius transformation. We have $h(0) = 0$ and $h(1) = \infty$, $h(-1) = 1/2$, $h(i) = (1-i)/2$ so that it maps R to the halph plane to the right of the line through $1/2$ and $(1-i)/2$. This plane is simply connected and does not contain 0 so that there is a well defined and analytic branch of the logartithm on $h(R)$.

On R we have:

$$f'(z) = \frac{h'(z)}{h(z)} = \frac{1}{z(1-z)} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+2}} \tag{9}$$

so that

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{z^n} \tag{10}$$

5. (12 points) Let D be the crescent shaped region between the circles $C_1 = \{|z| = 1\}$ and $C_2 = \{|z - 1/2| = 1/2\}$. Find an analytic function f on D such that $\Re f(z) = 1$ for $z \in C_1$ and $\Re f(z) = 0$ for $z \in C_2$. (**Hint**: use the Möbius transormation that maps D on a vertical strip.)

Solution: Let $h(z) = 1/(z-1)$. We have $h(C_1) = \{z \,|\, \Re(z) = -1/2\}$ while $h(C_2) = \{z \,|\, \Re(z) = -1\}$. Let $g(z) = 2(z+1)$. Clearly $\Re g(z) = 1$ for $z \in h(C_1)$ and $\Re g(z) = 0$ for $z \in h(C_2)$. Thus we can take

$$f(z) = g\left(h(z)\right) = 2\left(\frac{1}{z-1} + 1\right) = \frac{2z}{1-z} \tag{11}$$

6. (15 points) Let m and n be positive integer. Given a_i, $i = 1, \ldots, n$ and b_j, $j = 1, \ldots, m$ in $\mathbb{C}$ with $a_i \neq a_j$ for $i \neq j$, define:

$$A = \sum_{i=1}^{n} \frac{\prod_{j=1}^{m}(a_i - b_j)}{\prod_{k=1,\, k\neq i}^{n}(a_i - a_k)} \tag{12}$$

Prove that if $n \geq m+2$ than $A = 0$. (**Hint:** Consider a suitable rational function $f = P/Q$ constructed from the a_i and b_j and integrate it on a suitable cirle.)

Solution: Let $P(z) = \prod_{j=1}^{m}(z - b_j)$, $Q(z) = \prod_{i=1}^{n}(z - a_i)$ and $\gamma_R = \{z \mid |z| = R\}$ with $R > \max_i |a_i|$. Clearly

$$\int_{\gamma_R} \frac{P(z)}{Q(z)} dz = 2\pi i A \tag{13}$$

On the hand, since $n \geq m+2$, for $|z|$ large enough we have:

$$\left| \frac{P(z)}{Q(z)} \right| \leq \frac{C}{|z|^2} \tag{14}$$

for a suitable contant C. Thus

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{P(z)}{Q(z)} dz = 0. \tag{15}$$

7. Let f be an entire function with f not identically 0.

   (a) (5 points) Show that the set $Z = \{a \mid f(a) = 0\}$ is at most countable.

   Solution: Let $D_n = \{z \mid |z| \le n\}$ and $Z_n = \{a \in D_n \mid f(a) = 0\}$. Since D_n is compact we have that Z_n is finite. Moreover $Z = \bigcup_{n=0}^{\infty} Z_n$ so that Z is at most countable.

   (b) (10 points) Suppose that for every $a \in \mathbb{C}$ the series expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \tag{16}$$

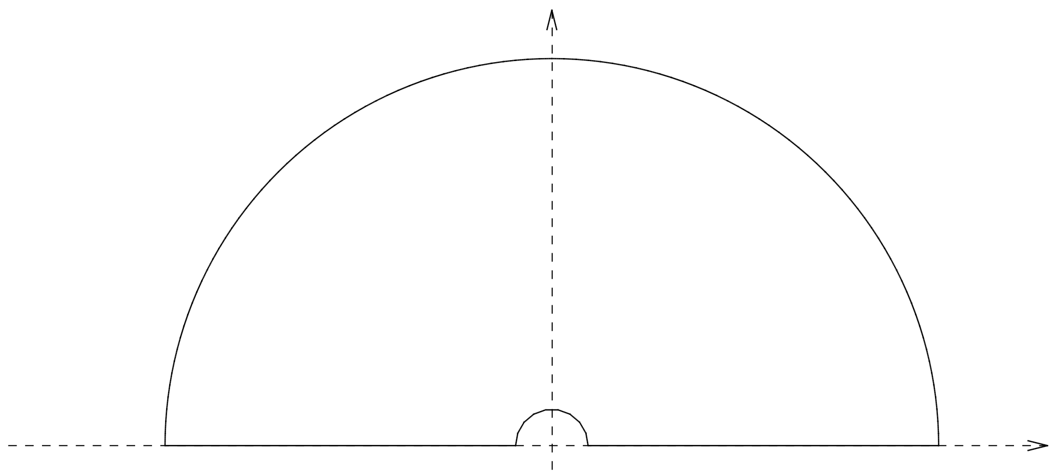
   has at least one 0 coefficient. Prove that f is a polynomial.

   Solution: For every n the n-th derivative $f^{(n)}$ of f is entire. Thus if $f^{(n)}$ is not identically 0 the set $Z^{(n)} = \{a \mid f^{(n)}(a) = 0\}$ is at most countable. Clearly $Z^{(n)} = \{a \mid a_n = 0\}$. It follows that the set $Z^{(\infty)} = \bigcup_{n=0}^{\infty} Z^{(n)} = \{a \mid a_n = 0 \text{ for at leat one } n\}$ is at most countable. Thus there must be n such that $f^{(n)} \equiv 0$ so that f is a polynomial.

8. (10 points) Show that

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}dx = \frac{\pi}{2}(1-e^{-1}) \tag{17}$$

Solution: Consider the path γ in the figure calling γ_R the large semicircle of radius R and γ_r the small one of radius r.



Consider the integral

$$I = \int_\gamma \frac{e^{iz}}{z(z^2+1)}dz \tag{18}$$

Since the integrand is meromorphic with one simple pole in $z = i$ we have

$$I = 2\pi i Res\left(\frac{e^{iz}}{x(z^2+1)}, i\right) = -i\pi e^{-1} \tag{19}$$

Observe that

$$\int_{\gamma_R} \frac{e^{iz}}{z(z^2+1)}dz \longrightarrow_{R\to\infty} 0 \tag{20}$$

while

$$\int_{\gamma_r} \frac{e^{iz}}{z(z^2+1)}dz \longrightarrow_{r\to 0} i\pi \tag{21}$$

Finally

$$\int_{-R}^{-r} \frac{e^{ix}}{x(x^2+1)}dx + \int_{-r}^{-R} \frac{e^{ix}}{x(x^2+1)}dx = 2i\int_r^R \frac{\sin(x)}{x(x^2+1)}dx \tag{22}$$

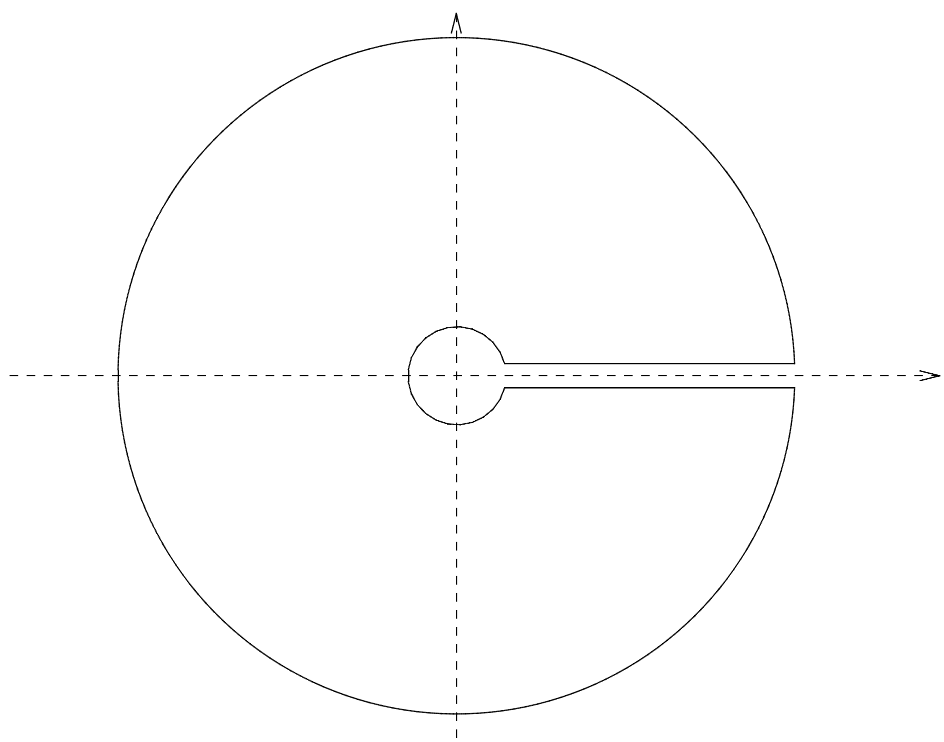
Collecting all terms we get the result.

9. (15 points) Show that

$$\int_0^\infty \frac{\sin(\sqrt{x})}{(4x^2+1)}dx = \frac{\pi}{2}\sin\left(\frac{1}{2}\right)\exp\left(-\frac{1}{2}\right) \tag{23}$$

(*Hint*: check example V.2.12.)

Solution: Consider the path γ in the figure calling γ_R the large circle of radius R, γ_r the small one of radius r, γ_+ the straight segment with positive immaginary part and γ_- the one with positive immaginary part.



Cosider the integral

$$I = \int_\gamma \frac{e^{i\sqrt{z}}}{(4z^2+1)}dz \tag{24}$$

where $\sqrt{z}$ is defined for $0 < \arg(z) < 2\pi$ as $\sqrt{z} = \sqrt{|z|}e^{i\arg(z)/2}$ Since there are two pole of the integarnd in the domain inside γ we get:

$$\begin{aligned} I &= 2\pi i Res\left(\frac{e^{i\sqrt{z}}}{(4z^2+1)}, \frac{i}{2}\right) + 2\pi i Res\left(\frac{e^{i\sqrt{z}}}{(4z^2+1)}, -\frac{i}{2}\right) = \\ &= 2\pi i\left(\frac{e^{\frac{-1-i}{2}}}{4i} - \frac{e^{\frac{-1+i}{2}}}{4i}\right) = i\pi\sin\left(\frac{1}{2}\right)\exp\left(-\frac{1}{2}\right) \end{aligned} \tag{25}$$

Since the integrand is bounded near the origin we have

$$\int_{\gamma_r} \frac{e^{i\sqrt{z}}}{(4z^2+1)} dz \longrightarrow_{r\to 0} 0 \tag{26}$$

Oberve that for $z \in \gamma_R$ we have $\Im\sqrt{z} > 0$. Thus, for $z \in \gamma_R$, $|e^{i\sqrt{z}}| \leq 1$ so that

$$\int_{\gamma_R} \frac{e^{i\sqrt{z}}}{(4z^2+1)} dz \longrightarrow_{R\to \infty} \infty \tag{27}$$

Finally observe that on γ_+, $\sqrt{z} = \sqrt{x}$ while, on γ_-, $\sqrt{z} = -\sqrt{x}$ so that

$$\begin{aligned} &\int_{\gamma_+} \frac{e^{i\sqrt{z}}}{(4z^2+1)} dz + \int_{\gamma_+} \frac{e^{i\sqrt{z}}}{(4z^2+1)} dz = \\ &\int_r^R \frac{e^{i\sqrt{x}}}{(4x^2+1)} dx + \int_R^r \frac{e^{-i\sqrt{x}}}{(4x^2+1)} dx = 2i \int_r^R \frac{\sin(\sqrt{z})}{(4x^2+1)} dx \end{aligned} \tag{28}$$

Taking the limits and collecting the factors we get the result.

10. (10 points) Let f be analytic on Ω open and connected. Prove that if $a \in \Omega$ and $B(a, r) \subset \Omega$ then

$$f(a) = \frac{1}{\pi r^2} \int_{B(a,r)} f(x, y) dx\, dy \tag{29}$$

where, with a slight abuse of notation, we have used the same symbol for the function f thought as a function $\mathbb{C} \to \mathbb{C}$ or $\mathbb{R}^2 \to \mathbb{R}^2$.

Solution: Let $\gamma_s = \{a + se^{it},\, 0 \le t \le 2\pi\}$ where $s \le r$. We have:

$$f(a) = \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(z)}{z - a} dz = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + se^{it}\right) \tag{30}$$

We can now moltiply both side by s and integrate over s from 0 to r. We get:

$$\frac{r^2}{2} f(a) = \frac{1}{2\pi} \int_0^r ds \int_0^{2\pi} f(\Re(a) + s\cos(t), \Im(a) + s\sin(t)) s ds\, dt \tag{31}$$

The thesis follows immediately by calling $x = \Re(a) + s\cos(t)$ and $y = \Im(a) + s\sin(t)$.

11. (10 points) Let f and g be two analytic function from $D = \{z \mid |z| < 1\}$ to $\Omega \subset \mathbb{C}$. Assume that f and g are invertible with analytic inverse. Suppose that there are two points z_1, $z_2 \in D$, $z_1 \neq z_2$, such that $f(z_1) = g(z_1)$ and $f(z_2) = g(z_2)$. Show that $f \equiv g$. (**Hint**: use one of the consequences of Schwarz's Lemma. You may first assume that $z_1 = 0$ and then get the general case.)

Solution: Since g is invertible with analytic inverse we have that $h(z) = g^{-1}(f(z))$ is analytic on D. Moreover h is invertible with analytic inverse. From theorem VI.2.5 it follows that

$$h(z) = c\frac{z-a}{1-\bar{a}z} \tag{32}$$

with $|c| = 1$ and $|a| \leq 1$. From the hypothesis we have that $h(z_1) = z_1$ and $h(z_2) = z_2$. Assume first that $z_1 = 0$. In this case $h(0) = 0$ implies that $a = 0$ and $h(z_2) = z_2$ implies $c = 1$. Thus $h(z) = z$ and $g(z) = f(z)$ for every z.

In the general case let

$$l(z) = \frac{z + z_1}{1 + \bar{z}_1 z}. \tag{33}$$

Clearly $\tilde{h}(z) = l^{-1}(h(l(z)))$ satisfy $\tilde{h}(0) = 0$ and $\tilde{h}(l^{-1}(z_2)) = l^{-1}(z_2)$ so that $\tilde{h}(z) = z$. Again we have $h(z) = z$.