You can use your book and notes. No laptop or wireless devices allowed. Write clearly and try to make your arguments as linear and simple as possible. The complete solution of one exercise will be considered more that two half solutions.

When returning your Exam, you must return also this page, signed. Thanks.

To solve the Exam problems, I have not collaborated with anyone or used any source except class notes and the textbook.

Name: ______________________________

Question:	1	2	3	4	5	Total
Points:	20	30	50	40	20	160
Score:						

1. (20 points) Let f be a Lipschitz continuous function from $\mathbb{R}^p$ to $\mathbb{R}^n$, $p < n$. This mean that there exists K such that

$$|f(x) - f(y)| \leq K|x - y|$$

for every $x, y \in \mathbb{R}^p$. Show that $m(f(\mathbb{R}^p)) = 0$ where m is the Lebesgue measure on $\mathbb{R}^n$. (**Hint**: start restricting f to the compact set $[-N, N]^p$ and cover it with balls of radius ϵ.)

2. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two measurable spaces. If μ is a finite signed measure on $(X, \mathcal{M})$ define $F_*\mu(B) = \mu(F^{-1}(B))$ for $B \in \mathcal{N}$.

   (a) (10 points) Show that μ is a finite signed measure.

(b) (20 points) Let f be a measurable function from Y to $\mathbb{R}$ such that $f \circ F$ is integrable. Show that

$$\int f dF_* \mu = \int f \circ F d\mu.$$

(**Hint**: assume first f is a characteristic function, then a simple function, then ...)

(c) Suppose that F is one-to-one and F^{-1} is measurable. Let μ and ν be two finite signed measure on $(X, \mathcal{M})$ such that

$$\frac{d\mu}{d\nu} = g.$$

Show that

$$\frac{dF_*\mu}{dF_*\nu} = g \circ F^{-1}.$$

(**Hint**: aplly point b) to the characteristic function of a measurable set E.)

3. Let F be a measurable function from $[a, b]$ to $\mathbb{R}$. Define

$$\operatorname*{ess\,sup}_{[a,b]} f = \inf \{L \mid m(\{x \mid f(x) > L\}) = 0\}$$

where m is the Lebesgue measure.

(a) (10 points) Show that for $f > 0$

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \operatorname*{ess\,sup}_{[a,b]} f$$

(b) (20 points) Show that if f is continuous then

$$\operatorname*{ess\,sup}_{[a,b]} f = \sup_{[a,b]} f$$

(c) (20 points) Does there exists $f \in L^1([0,1])$ such that

$$\operatorname*{ess\,sup}_{[a,b]} f = +\infty$$

for every $0 < a < b < 1$?

4. Let f be an integrable function from $\mathbb{R}^+$ to $\mathbb{R}$. Let

$$g_n(x) = \sum_{n=0}^{n} f(x+n).$$

   (a) (20 points) Show that there exists $g \in L^1([0,1])$ such that $\lim_{n\to\infty} g_n = g$ in $L^1([0,1])$. (**Hint**: show that g_n is Cauchy in $L^1([0,1])$.)

(b) (20 points) Show that if also $xf(x)$ is integrable then there exists $g \in L^1(\mathbb{R})$ such that $\lim_{n\to\infty} g_n = g$ in $L^1(\mathbb{R})$.

(c) Does $g_n(x)$ converge to $g(x)$ for almost every $x \in [0, 1]$? and for almost every $x \in \mathbb{R}^+$? (**Hint**: assume first that $f > 0$.)

5. (20 points) Let $E \subset \mathbb{R}$ be a measurable set. Show that $m(E) = 0$ iff there exists $f \in L^1(\mathbb{R})$ such that

$$\lim_{r \to 0} \frac{1}{r} \int_{x-r}^{x+r} f(y)dy = \infty \qquad \forall x \in E.$$

(**Hint**: consider a sequnce of open set U_n with $U_0 \supset U_1 \supset \cdots \supset U_n \supset \cdots \supset E$ and $m(U_n) \leq 1/n^2$ ...)