Math 1553 Worksheet: Lines and planes in $\mathbb{R}^n$, §1.1, §1.2

Solutions

1. For each equation, determine whether the equation is linear or non-linear, and circle your answer.

a) $3x_1 + \sqrt{x_2} = 4$ Linear Not linear

b) $e^{\pi}x + \ln(13)y = \sqrt{2} - z$ Linear Not linear

Solution.

a) Not linear. The $\sqrt{x_2}$ term makes it non-linear.

b) Linear. Note e^{π} and $\sqrt{2}$ are just real numbers. Don't be misled by the appearance of the natural logarithm: $\ln(13)$ is just the coefficient for y.

If the second term had been $\ln(13y)$ instead of $\ln(13)y$, then y would have been inside the logarithm and the equation would have been non-linear.

2. Consider the following three planes, where we use (x, y, z) to denote points in $\mathbf{R}^3$:

$$2x + 4y + 4z = 1$$

$$2x + 5y + 2z = -1$$

$$y + 3z = 8.$$

Find all points where the planes intersect. In other words, find all solutions to the system of three linear equations given above.

Solution.

We can isolate z in the third equation using algebra, but it is probably best to do using an augmented matrix and elementary row operations.

$$\left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 2 & 5 & 2 & -1 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_3 = R_3 - R_2} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{array}\right).$$

The last line is $5z = 10$, so $z = 2$. From here, we can back-substitute or continue doing row operations to finish. Below, we give solutions using each method.

Using Substitution: The second row gives $y - 2z = -2$, so $y - 2(2) = -2$ so $y = 2$.

The first equation is $2x + 4(2) + 4(2) = 1$, so $2x = -15$, thus $x = -\dfrac{15}{2}$.

We have found that the planes intersect at the point $\left(-\dfrac{15}{2},\ 2,\ 2\right)$.

Using more row operations: We eliminate the nonzero terms above the diagonal on the left-hand side of the augmented matrix.

$$\left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{array}\right) \xrightarrow{R_3=\frac{R_3}{5}} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_2=R_2+2R_3} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1=R_1-4R_3} \left(\begin{array}{ccc|c} 2 & 4 & 0 & -7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-4R_2} \left(\begin{array}{ccc|c} 2 & 0 & 0 & -15 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1=\frac{R_1}{2}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -15/2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

This final augmented matrix says $x=-\dfrac{15}{2},\ \ y=2,\ \ z=2.$

3. For each of the following, answer true or false. Justify your answer.

a) Every system of linear equations has at least one solution.

b) There is a system of linear equations that has exactly 5 solutions.

Solution.

a) False. For example, the system

$$\begin{aligned} x+y&=5 \\ x+y&=1. \end{aligned}$$

b) False. There are only three possibilities: no solutions, exactly one solution, or infinitely many solutions. In section 1.2, we will see why.

4. **a)** Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

b) Which entries are the pivots? Which are the pivot columns?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

Solution.

The first is in reduced row echelon form; the second is in row echelon form. The pivots are in red; the other entries in the pivot columns are in blue.