

Section 1.4

The Matrix Equation $Ax = b$

Matrix \times Vector

the first number is
the number of rows

the second number is
the number of columns

Let A be an $m \times n$ matrix

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix} \quad \text{with columns } v_1, v_2, \dots, v_n$$

Definition

The **product** of A with a vector x in \mathbf{R}^n is the linear combination

$$Ax = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \stackrel{\text{def}}{=} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

this means the equality is a definition

The output is a vector in \mathbf{R}^m .

these must be equal

Note that the number of **columns** of A has to equal the number of **rows** of x .

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Matrix Equations

An example

Question

Let v_1, v_2, v_3 be vectors in \mathbf{R}^3 . How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

Answer: Let A be the matrix with columns v_1, v_2, v_3 , and let x be the vector with entries $2, 3, -4$. Then

$$Ax = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 2v_1 + 3v_2 - 4v_3,$$

so the vector equation is equivalent to the matrix equation

$$Ax = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}.$$

Matrix Equations

In general

Let v_1, v_2, \dots, v_n , and b be vectors in \mathbf{R}^m . Consider the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

It is equivalent to the **matrix equation**

$$Ax = b$$

where

$$A = \left(\begin{array}{c|c|c|c} | & | & \cdots & | \\ v_1 & v_2 & & v_n \\ | & | & & | \end{array} \right) \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, if A is any $m \times n$ matrix, then

$$Ax = b \quad \text{is equivalent to the} \quad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$$

vector equation

where v_1, \dots, v_n are the columns of A , and x_1, \dots, x_n are the entries of x .

We now have *four* equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:

$$\begin{aligned}2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5\end{aligned}$$

2. As an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array} \right)$$

3. As a vector equation ($x_1 v_1 + \dots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation ($Ax = b$):

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

We will move back and forth freely between these over and over again, for the rest of the semester. Get comfortable with them now!

In particular, *all four have the same solution set.*

Matrix \times Vector

Another way

Definition

A **row vector** is a matrix with one row. The product of a row vector of length n and a (column) vector of length n is

$$(a_1 \quad \cdots \quad a_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \stackrel{\text{def}}{=} a_1 x_1 + \cdots + a_n x_n.$$

This is a scalar.

If A is an $m \times n$ matrix with rows r_1, r_2, \dots, r_m , and x is a vector in \mathbf{R}^n , then

$$Ax = \begin{pmatrix} -r_1- \\ -r_2- \\ \vdots \\ -r_m- \end{pmatrix} x = \begin{pmatrix} r_1 x \\ r_2 x \\ \vdots \\ r_m x \end{pmatrix}$$

This is a vector in \mathbf{R}^m (again).

Matrix \times Vector

Both ways

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} (4\ 5\ 6) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ (7\ 8\ 9) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 \\ 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Note this is the same as before:

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\ 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Now you have *two* ways of computing Ax .

In the second, you calculate Ax one entry at a time.

The second way is usually the most convenient, but we'll use both.

Spans and Solutions to Equations

Let A be a matrix with columns v_1, v_2, \dots, v_n :

$$A = \left(\begin{array}{c|c|c|c} | & | & \cdots & | \\ v_1 & v_2 & & v_n \\ | & | & & | \end{array} \right)$$

Very Important Fact That Will Appear on Every Midterm and the Final

$Ax = b$ has a solution

$$\iff \text{there exist } x_1, \dots, x_n \text{ such that } A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

"if and only if"

$$\iff \text{there exist } x_1, \dots, x_n \text{ such that } x_1 v_1 + \cdots + x_n v_n = b$$

$$\iff b \text{ is a linear combination of } v_1, \dots, v_n$$

$$\iff b \text{ is in the span of the columns of } A.$$

The last condition is geometric.

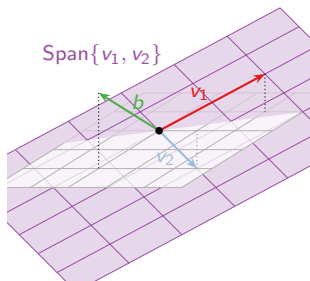
Spans and Solutions to Equations

Example

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

[interactive]



Columns of A:

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Target vector:

$$b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A ? It sure doesn't look like it.

Conclusion: $Ax = b$ is *inconsistent*.

Spans and Solutions to Equations

Example, continued

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's check by solving the matrix equation using row reduction.

The first step is to put the system into an augmented matrix.

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array} \right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

The last equation is $0 = 1$, so the system is *inconsistent*.

In other words, the matrix equation

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

has no solution, as the picture shows.

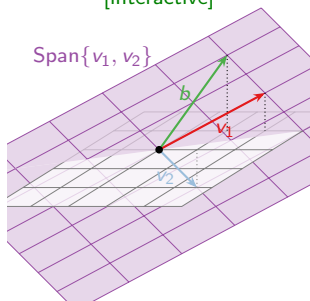
Spans and Solutions to Equations

Example

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

[interactive]



Columns of A:

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Target vector:

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A ? It looks like it: in fact,

$$b = 1v_1 + (-1)v_2 \implies x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Spans and Solutions to Equations

Example, continued

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's do this systematically using row reduction.

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array} \right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right)$$

This gives us

$$x_1 = 1 \quad x_2 = -1.$$

This is consistent with the picture on the previous slide:

$$1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{or} \quad A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

When Solutions Always Exist

Here are criteria for a linear system to *always* have a solution.

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution for all b in \mathbf{R}^m .
2. The span of the columns of A is all of \mathbf{R}^m .
3. A has a pivot in each row.

recall that this means
that for given A , either they're
all true, or they're all false

Why is (1) the same as (2)? This was the Very Important box from before.

Why is (1) the same as (3)? If A has a pivot in each row then its reduced row echelon form looks like this:

$$\begin{pmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix} \quad \text{and } (A | b) \quad \begin{pmatrix} 1 & 0 & * & 0 & * & | & * \\ 0 & 1 & * & 0 & * & | & * \\ 0 & 0 & 0 & 1 & * & | & * \end{pmatrix}.$$

reduces to this:

There's no b that makes it inconsistent, so there's always a solution. If A doesn't have a pivot in each row, then its reduced form looks like this:

$$\begin{pmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and this can be} \quad \begin{pmatrix} 1 & 0 & * & 0 & * & | & 0 \\ 0 & 1 & * & 0 & * & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 16 \end{pmatrix}.$$

made
inconsistent:

When Solutions Always Exist

Continued

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution for all b in \mathbf{R}^m .
2. The span of the columns of A is all of \mathbf{R}^m .
3. A has a pivot in each row.

In the following demos, the red region is the span of the columns of A . This is the same as the set of all b such that $Ax = b$ has a solution.

[example where the criteria are satisfied]

[example where the criteria are not satisfied]

Properties of the Matrix–Vector Product

Let c be a scalar, u, v be vectors, and A a matrix.

▶ $A(u + v) = Au + Av$

▶ $A(cv) = cAv$

See Lay, §1.4, Theorem 5.

For instance, $A(3u - 7v) = 3Au - 7Av$.

Consequence: If u and v are solutions to $Ax = 0$, then so is every vector in $\text{Span}\{u, v\}$. Why?

$$\begin{cases} Au = 0 \\ Av = 0 \end{cases} \implies A(xu + yv) = xAu + yAv = x0 + y0 = 0.$$

(Here 0 means the zero vector.)

Important

The set of solutions to $Ax = 0$ is a span.

Summary

- ▶ We have four equivalent ways of writing a system of linear equations:
 1. As a system of equations.
 2. As an augmented matrix.
 3. As a vector equation.
 4. As a matrix equation $Ax = b$.
- ▶ $Ax = b$ is consistent if and only if b is in the span of the columns of A . The latter condition is geometric: you can draw pictures of it.
- ▶ $Ax = b$ is consistent for all b in \mathbf{R}^m if and only if the columns of A span \mathbf{R}^m .