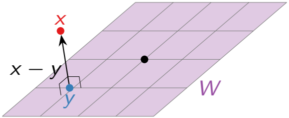
Section 6.2/6.3

Orthogonal Projections

Best Approximation

Suppose you measure a data point x which you know for theoretical reasons must lie on a subspace W.



Due to measurement error, though, the measured x is not actually in W. Best approximation: y is the *closest* point to x on W.

How do you know that y is the closest point? The vector from y to x is orthogonal to W: it is in the *orthogonal complement* $W^\perp$.

Orthogonal Decomposition

Recall: If W is a subspace of $\mathbf{R}^n$, its **orthogonal complement** is

$$W^\perp = \{v \text{ in } \mathbf{R}^n \mid v \text{ is perpendicular to every vector in } W\}$$

Theorem

Every vector x in $\mathbf{R}^n$ can be written as

$$x = x_W + x_{W^\perp}$$

for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.

The equation $x = x_W + x_{W^\perp}$ is called the **orthogonal decomposition** of x (with respect to W).

The vector x_W *is the closest vector to* x *on* W.

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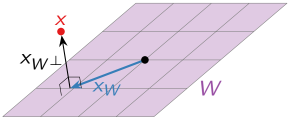


Theorem

Every vector x in $\mathbf{R}^n$ can be written as

$$x = x_W + x_{W^\perp}$$

for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.

Why?

Uniqueness: suppose $x = x_W + x_{W^\perp} = x'_W + x'_{W^\perp}$ for x_W, x'_W in W and $x_{W^\perp}, x'_{W^\perp}$ in $W^\perp$. Rewrite:

$$x_W - x'_W = x'_{W^\perp} - x_{W^\perp}.$$

The left side is in W, and the right side is in $W^\perp$, so they are both in $W \cap W^\perp$. But the only vector that is perpendicular to itself is the zero vector! Hence

$$0 = x_W - x'_W \implies x_W = x'_W$$
$$0 = x_{W^\perp} - x'_{W^\perp} \implies x_{W^\perp} = x'_{W^\perp}$$

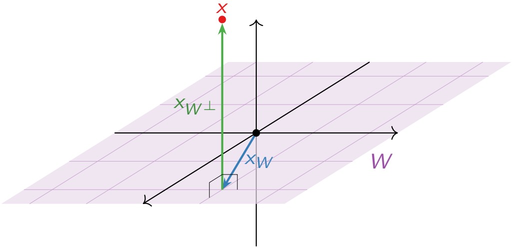
Existence: We will compute the orthogonal decomposition later using orthogonal projections.

Let W be the xy-plane in $\mathbf{R}^3$. Then $W^\perp$ is the z-axis.

$$x = \begin{pmatrix}1\\2\\3\end{pmatrix} \implies x_W = \begin{pmatrix}1\\2\\0\end{pmatrix} \qquad x_{W^\perp} = \begin{pmatrix}0\\0\\3\end{pmatrix}.$$

$$x = \begin{pmatrix}a\\b\\c\end{pmatrix} \implies x_W = \begin{pmatrix}a\\b\\0\end{pmatrix} \qquad x_{W^\perp} = \begin{pmatrix}0\\0\\c\end{pmatrix}.$$

This is just decomposing a vector into a "horizontal" component (in the xy-plane) and a "vertical" component (on the z-axis).



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Problem: Given x and W, how do you compute the decomposition $x = x_W + x_{W^\perp}$?

Observation: It is enough to compute x_W, because $x_{W^\perp} = x - x_W$.

First we need to discuss orthogonal sets.

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if, in addition, each vector is a unit vector.

Lemma

An orthogonal set of vectors is linearly independent. Hence it is a basis for its span.

Suppose $\{u_1, u_2, \ldots, u_m\}$ is orthogonal. We need to show that the equation

$$c_1 u_1 + c_2 u_2 + \cdots + c_m u_m = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_m = 0$.

$$0 = u_1 \cdot (c_1 u_1 + c_2 u_2 + \cdots + c_m u_m) = c_1(u_1 \cdot u_1) + 0 + 0 + \cdots + 0.$$

Hence $c_1 = 0$. Similarly for the other c_i.

Example: $\mathcal{B} = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\}$ is an orthogonal set. Check:

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-2\\1\end{pmatrix} = 0 \qquad \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix} = 0 \qquad \begin{pmatrix}1\\-2\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix} = 0.$$

Example: $\mathcal{B} = \{e_1, e_2, e_3\}$ is an orthogonal set. Check:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = 0 \qquad \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} = 0 \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} = 0.$$

Example: Let $x = \binom{a}{b}$ be a nonzero vector, and let $y = \binom{-b}{a}$. Then $\{x, y\}$ is an orthogonal set:

$$\begin{pmatrix}a\\b\end{pmatrix} \cdot \begin{pmatrix}-b\\a\end{pmatrix} = -ab + ab = 0.$$

Orthogonal Projections

Definition

Let W be a subspace of $\mathbf{R}^n$, and let $\{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\boxed{\operatorname{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.}$$

This is a vector in W because it is in $\operatorname{Span}\{u_1, u_2, \ldots, u_m\}$.

Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then

$$\boxed{x_W = \operatorname{proj}_W(x) \qquad \text{and} \qquad x_{W^\perp} = x - \operatorname{proj}_W(x).}$$

In particular, $\operatorname{proj}_W(x)$ is the closest point to x in W.

Why? Let $y = \operatorname{proj}_W(x)$. We need to show that $x - y$ is in $W^\perp$. In other words, $u_i \cdot (x - y) = 0$ for each i. Let's do u_1:

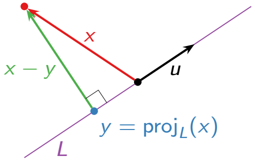
$$u_1 \cdot (x - y) = u_1 \cdot \left(x - \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i \right) = u_1 \cdot x - \frac{x \cdot u_1}{u_1 \cdot u_1}(u_1 \cdot u_1) - 0 - \cdots = 0.$$

Orthogonal Projection onto a Line

The formula for orthogonal projections is simple when W is a *line*.

Let $L = \text{Span}\{u\}$ be a line in $\mathbf{R}^n$, and let x be in $\mathbf{R}^n$. The orthogonal projection of x onto L is the point

$$\text{proj}_L(x) = \frac{x \cdot u}{u \cdot u}\, u.$$



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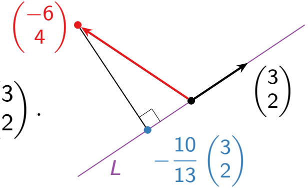
Example: Compute the orthogonal projection of $x = \binom{-6}{4}$ onto the line L spanned by $u = \binom{3}{2}$.

$$y = \text{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u = \frac{-18+8}{9+4}\begin{pmatrix}3\\2\end{pmatrix} = -\frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix}.$$



Orthogonal Projection onto a Plane

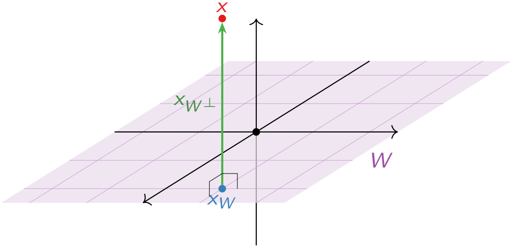
Easy example

What is the projection of $x = \begin{pmatrix}1\\2\\3\end{pmatrix}$ onto the xy-plane?

Answer: The xy-plane is $W = \text{Span}\{e_1, e_2\}$, and $\{e_1, e_2\}$ is an orthogonal basis.

$$x_W = \text{proj}_W \begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{x \cdot e_1}{e_1 \cdot e_1} e_1 + \frac{x \cdot e_2}{e_2 \cdot e_2} e_2 = \frac{1 \cdot 1}{1^2} e_1 + \frac{1 \cdot 2}{1^2} e_2 = \begin{pmatrix}1\\2\\0\end{pmatrix}.$$

So this is the same projection as before.



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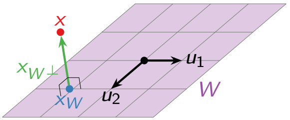
What is the projection of $x = \begin{pmatrix} -1.1 \\ 1.4 \\ 1.45 \end{pmatrix}$ onto $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.1 \\ -.2 \end{pmatrix} \right\}$?

Answer: The basis is orthogonal, so

$$x_W = \text{proj}_W \begin{pmatrix} -1.1 \\ 1.4 \\ 1.45 \end{pmatrix} = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \frac{(-1.1)(1)}{1^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{(1.4)(1.1) + (1.45)(-.2)}{1.1^2 + (-.2)^2} \begin{pmatrix} 0 \\ 1.1 \\ -.2 \end{pmatrix}$$

This turns out to be equal to $u_2 - 1.1u_1$.



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First we restate the property we've been using all along.

Let x be a vector and let $x = x_W + x_{W^\perp}$ be its orthogonal decomposition with respect to a subspace W. The following vectors are the same:

- x_W
- $\mathrm{proj}_W(x)$
- The closest vector to x on W

We can think of orthogonal projection as a *transformation*:

$$\mathrm{proj}_W : \mathbf{R}^n \longrightarrow \mathbf{R}^n \qquad x \mapsto \mathrm{proj}_W(x).$$

Theorem

Let W be a subspace of $\mathbf{R}^n$.

1. proj_W is a *linear* transformation.
2. For every x in W, we have $\mathrm{proj}_W(x) = x$.
3. For every x in $W^\perp$, we have $\mathrm{proj}_W(x) = 0$.
4. The range of proj_W is W and the null space of proj_W is $W^\perp$.

Poll

Let W be a subspace of $\mathbf{R}^n$ with $W \neq \{0\}$ and $W \neq \mathbf{R}^n$.

Poll

Let A be the matrix for proj_W. What is/are the eigenvalue(s) of A?

A. 0 B. 1 C. -1 D. $0, 1$ E. $1, -1$ F. $0, -1$ G. $-1, 0, 1$

The 1-eigenspace is W.

The 0-eigenspace is $W^\perp$.

We have $\dim W + \dim W^\perp = n$, so that gives n linearly independent eigenvectors already.

So the answer is D.

What is the matrix for $\mathrm{proj}_W \colon \mathbf{R}^3 \to \mathbf{R}^3$, where

$$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}?$$

Answer: Recall how to compute the matrix for a linear transformation:

$$A = \begin{pmatrix} | & | & | \\ \mathrm{proj}_W(e_1) & \mathrm{proj}_W(e_2) & \mathrm{proj}_W(e_3) \\ | & | & | \end{pmatrix}.$$

We compute:

$$\mathrm{proj}_W(e_1) = \frac{e_1 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_1 \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/6 \\ 1/3 \\ -1/6 \end{pmatrix}$$

$$\mathrm{proj}_W(e_2) = \frac{e_2 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_2 \cdot u_2}{u_2 \cdot u_2} u_2 = 0 + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$\mathrm{proj}_W(e_3) = \frac{e_3 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_3 \cdot u_2}{u_2 \cdot u_2} u_2 = -\frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/6 \\ 1/3 \\ 5/6 \end{pmatrix}$$

Therefore $A = \begin{pmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{pmatrix}$.

Let W be an m-dimensional subspace of $\mathbf{R}^n$, let $\text{proj}_W \colon \mathbf{R}^n \to W$ be the projection, and let A be the matrix for proj_W.

Fact 1: A is diagonalizable with eigenvalues 0 and 1; it is similar to the diagonal matrix with m ones and $n - m$ zeros on the diagonal.

Why? Let $v_1, v_2, \ldots, v_m$ be a basis for W, and let $v_{m+1}, v_{m+2}, \ldots, v_n$ be a basis for $W^\perp$. These are (linearly independent) eigenvectors with eigenvalues 1 and 0, respectively, and they form a basis for $\mathbf{R}^n$ because there are n of them.

Example: If W is a plane in $\mathbf{R}^3$, then A is similar to projection onto the xy-plane:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Fact 2: $A^2 = A$.

Why? Projecting twice is the same as projecting once:

$$\text{proj}_W \circ \text{proj}_W = \text{proj}_W \implies A \cdot A = A.$$

Coordinates with respect to Orthogonal Bases

Let W be a subspace with orthogonal basis $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$.

For x in W we have $\text{proj}_W(x) = x$, so

$$x = \text{proj}_W(x) = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_n}{u_n \cdot u_n}\, u_n.$$

This makes it easy to compute the $\mathcal{B}$-coordinates of x.

Corollary

Let W be a subspace with orthogonal basis $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$. Then

$$[x]_{\mathcal{B}} = \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \ \frac{x \cdot u_2}{u_2 \cdot u_2}, \ \ldots, \ \frac{x \cdot u_m}{u_m \cdot u_m} \right).$$

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Problem: Find the $\mathcal{B}$-coordinates of $x = \binom{0}{3}$, where

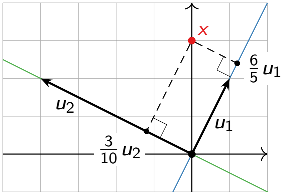
$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix} \right\}.$$

Old way:

$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 2 & 2 & 3 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 6/5 \\ 0 & 1 & 6/20 \end{array}\right) \implies [x]_{\mathcal{B}} = \begin{pmatrix} 6/5 \\ 6/20 \end{pmatrix}.$$

New way: note $\mathcal{B}$ is an *orthogonal* basis.

$$[x]_{\mathcal{B}} = \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2} \right) = \left(\frac{3 \cdot 2}{1^2 + 2^2}, \frac{3 \cdot 2}{(-4)^2 + 2^2} \right) = \left(\frac{6}{5}, \frac{3}{10} \right).$$



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What is the distance from e_1 to $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$?

Answer: The closest point on W to e_1 is $\text{proj}_W(e_1) = \begin{pmatrix} 5/6 \\ 1/3 \\ -1/6 \end{pmatrix}$.

The distance from e_1 to this point is

$$\begin{aligned} \text{dist}\big(e_1, \text{proj}_W(e_1)\big) &= \|(e_1)_{W^\perp}\| \\ &= \left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 5/6 \\ 1/3 \\ -1/6 \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} 1/6 \\ -1/3 \\ 1/6 \end{pmatrix} \right\| \\ &= \sqrt{(1/6)^2 + (-1/3)^2 + (1/6)^2} \\ &= \frac{1}{\sqrt{6}}. \end{aligned}$$

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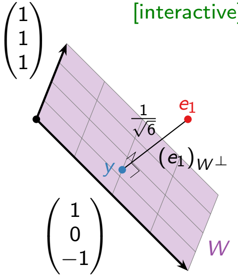

Summary

Let W be a subspace of $\mathbf{R}^n$.

- Any vector x in $\mathbf{R}^n$ can be written in a unique way as

$$x = x_W + x_{W^\perp}$$

  for x_W in W and $x_{W^\perp}$ in $W^\perp$. This is called its **orthogonal decomposition.**
- The vector x_W is the *closest point to* x *in* W: it is the *best approximation*.
- The *distance* from x to W is $\|x_{W^\perp}\|$.
- If you have an *orthogonal* basis $\{u_1, u_2, \ldots, u_m\}$ for W, then

$$x_W = \text{proj}_W(x) = \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_n}{u_n \cdot u_n}\, u_n.$$

  Hence $x_{W^\perp} = x - \text{proj}_W(x)$.
- If you have an *orthogonal* basis $\{u_1, u_2, \ldots, u_m\}$ for W, then

$$[x]_{\mathcal{B}} = \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \ldots, \frac{x \cdot u_m}{u_m \cdot u_m} \right).$$

- We can think of $\text{proj}_W \colon \mathbf{R}^n \to \mathbf{R}^n$ as a linear transformation. Its null space is $W^\perp$, and its range is W.
- The matrix A for proj_W is diagonalizable with eigenvalues 0 and 1. It is *idempotent*: $A^2 = A$.