Math 1553 Worksheet §1.2, §1.3

Solutions

1. Solve the following system of equations by putting an augmented matrix into reduced row echelon form. State which variables (if any) are free variables.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3. \end{aligned}$$

Solution.

$$\left(\begin{array}{ccc|c} \boxed{1} & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right) \xrightarrow{R_2=R_2+4R_1} \left(\begin{array}{ccc|c} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{3} & 6 & 3 \\ 0 & -3 & -6 & -3 \end{array}\right) \xrightarrow{R_3=R_3+R_2} \left(\begin{array}{ccc|c} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{3} & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1=R_1-3R_2} \left(\begin{array}{ccc|c} \boxed{1} & 0 & -5 & -2 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The variables x_1 and x_2 correspond to pivot columns, but x_3 is free. The solution set is a line in $\mathbf{R}^3$.

$$x_1 = -2 + 5x_3, \qquad x_2 = 1 - 2x_3, \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

2. Consider the system of linear equations

$$\begin{aligned} x + 2y &= 7 \\ 2x + y &= -2 \\ -x - y &= 4 \end{aligned}$$

Question: Does this system have a solution? If so, what is the solution set?

a) Formulate this question as an augmented matrix.

b) Formulate this question as a vector equation.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

Solution.

a) As an augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 1 & -2 \\ -1 & -1 & 4 \end{array}\right)$$

b) What are the solutions to the following vector equation?

$$x\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$$

c) There exists a solution if and only if $\begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

e) Row reducing yields

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right),$$

so there are no solutions. (This should be obvious from the picture in (d)).

3. Zander has challenged you to find his hidden treasure, located at some point (a, b, c). He has honestly guaranteed you that the treasure can be found by starting at the origin and taking steps in directions given by

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

By decoding Zander's message, you have discovered that the treasure's first and second entries are (in order) −4 and 3.

a) What is the treasure's full location?

b) Give instructions for how to find the treasure by only moving in the directions given by v_1, v_2, and v_3.

Solution.

a) We translate this problem into linear algebra. Let c be the final entry of the treasure's location. Since Zander has assured us that we can find the treasure using the three vectors we have been given, our problem is to find c so that $\begin{pmatrix} -4 \\ 3 \\ c \end{pmatrix}$ is a linear combination of v_1, v_2, and v_3 (in other words, find c so that the treasure's location is in in $\mathrm{Span}\{v_1, v_2, v_3\}$). We form an augmented matrix and find when it gives a consistent system.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & c \end{array}\right) \xrightarrow[R_3=R_3+2R_1]{R_2=R_2+R_1} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & c-8 \end{array}\right) \xrightarrow{R_3=R_3-3R_2} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & c-5 \end{array}\right).$$

This system will be consistent if and only if the right column is not a pivot column, so we need $c - 5 = 0$, or $c = 5$.

The location of the treasure is $(-4, 3, 5)$.

b) Getting to the point $(-4,3,5)$ using the vectors v_1, v_2, and v_3 is equivalent to finding scalars x_1, x_2, and x_3 so that

$$\begin{pmatrix}-4\\3\\5\end{pmatrix} = x_1\begin{pmatrix}1\\-1\\-2\end{pmatrix} + x_2\begin{pmatrix}5\\-4\\-7\end{pmatrix} + x_3\begin{pmatrix}-3\\1\\0\end{pmatrix}$$

We can rewrite this as

$$\begin{aligned} x_1 + 5x_2 - 3x_3 &= -4\\ -x_1 - 4x_2 + x_3 &= 3\\ -2x_1 - 7x_2 \phantom{{}+3x_3} &= 5. \end{aligned}$$

We put the matrix from part (a) into RREF.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4\\ 0 & 1 & -2 & -1\\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 = R_1 - 5R_2} \left(\begin{array}{ccc|c} 1 & 0 & 7 & 1\\ 0 & 1 & -2 & -1\\ 0 & 0 & 0 & 0 \end{array}\right).$$

Note x_3 is the only free variable, so:

$$x_1 = 1 - 7x_3, \qquad x_2 = -1 + 2x_3 \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

Since the system has infinitely many solutions, there are infinitely many ways to get to the treasure. If we choose the path corresponding to $x_3 = 0$, then $x_1 = 1$ and $x_2 = -1$, which means that we move 1 unit in the direction of v_1 and -1 unit in the direction of v_2. In equations:

$$\begin{pmatrix}-4\\3\\5\end{pmatrix} = \begin{pmatrix}1\\-1\\-2\end{pmatrix} - \begin{pmatrix}5\\-4\\-7\end{pmatrix} + 0\begin{pmatrix}-3\\1\\0\end{pmatrix}.$$