Math 1553 Supplement §1.7, 1.8, 1.9

Solutions

1. Justify why each of the following true statements can be checked without row reduction.

a) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \right\}$ is linearly independent.

b) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$ is linearly independent.

c) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly dependent.

Solution.

a) You can eyeball linear independence: if

$$x \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix} + z \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 3x \\ 3x + y\sqrt{2} \\ 4x + \pi z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then $x = 0$, so $y = z = 0$ too.

b) Since the first coordinate of $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ is nonzero, $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ cannot be in the span of $\left\{ \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$. And $\begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}$ is not in the span of $\left\{ \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$ because it is not a multiple. Hence the span gets bigger each time you add a vector, so they're linearly independent.

c) Any four vectors in $\mathbf{R}^3$ are linearly dependent; you don't need row reduction for that.

2. Let A be a 3×4 matrix with column vectors v_1, v_2, v_3, v_4. Suppose that $v_2 = 2v_1 - 3v_4$. Find one non-trivial solution to the equation $Ax = 0$.

Solution.

From the linear dependence condition we were given, we get

$$-2v_1 + v_2 + 3v_4 = 0.$$

This vector equation is just

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{so} \quad A\begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad \text{Thus, } x = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} \text{ is one solution.}$$

3. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

a) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (0, x)$.

b) JUST FOR FUN: Consider T : (Smooth functions) $\to$ (Smooth functions) given by $T(f) = f'$ (the derivative of f). Then T is not a transformation from any $\mathbf{R}^n$ to $\mathbf{R}^m$, but it is still *linear* in the sense that for all smooth f and g and all scalars c (by properties of differentiation we learned in Calculus 1):

$$T(f+g) = T(f) + T(g) \qquad ((f+g)' = f' + g')$$
$$T(cf) = cT(f) \qquad (cf)' = cf'.$$

Is T one-to-one?

Solution.

a) This is not onto. There is no (x, y, z) such that $T(x, y, z) = (1, 0)$. It is not one-to-one: for instance, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$.

b) T is not one-to-one. If T were one-to-one, then for any smooth function b, the equation $T(f) = b$ would have at most one solution. However, Note that if f and g are the functions $f(t) = t$ and $g(t) = t - 1$, then f and g are different functions but their derivatives are the same, so $T(f) = T(g)$. Therefore, T is not one-to-one.

We note again that this problem is just for fun! It is not within the scope of Math 1553. If you find it confusing, feel free to ignore it.

4. For each matrix A, describe what the associated matrix transformation T does to $\mathbf{R}^3$ geometrically.

$$\textbf{a)}\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution.

a) We compute

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}.$$

This is the reflection over the yz-plane.

b) We compute

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}.$$

This is projection onto the z-axis.