Math 1553 Supplement §2.8, 2.9

1. Find bases for the column space and the null space of

$$A = \begin{pmatrix} 0 & 1 & -3 & 1 & 0 \\ 1 & -1 & 8 & -7 & 1 \\ -1 & -2 & 1 & 4 & -1 \end{pmatrix}.$$

Solution.

The RREF of $\left(A \mid 0\right)$ is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 5 & -6 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

so x_3, x_4, x_5 are free, and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -5x_3 + 6x_4 - x_5 \\ 3x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for Nul A is $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

To find a basis for Col A, we use the pivot columns as they were written in the *original* matrix A, not its RREF. These are the first two columns:

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}.$$

2. Which of the following are subspaces of $\mathbf{R}^4$? Why or why not?

(a) $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x + y = 0 \text{ and } z + w = 0 \right\}$ (b) $W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; xy - zw = 0 \right\}$

Solution.

a) The null space of a 2×4 matrix is automatically a subspace of $\mathbf{R}^4$, and V is equal to the null space of the matrix below, so V is a subspace of $\mathbf{R}^4$:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Alternatively, we could verify the subspace properties directly if we wished. This is much more work!

(1) The zero vector is in V, since $0+0=0$ and $0+0=0$.

(2) If $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$ are in V. Compute $u+v = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \\ w_1+w_2 \end{pmatrix}$.
Are $(x_1+x_2)+(y_1+y_2)=0$ and $(z_1+z_2)+(w_1+w_2)=0$? Yes.

$$(x_1+x_2)+(y_1+y_2) = (x_1+y_1)+(x_2+y_2) = 0+0 = 0,$$

$$(z_1+z_2)+(w_1+w_2) = (z_1+w_1)+(z_2+w_2) = 0+0 = 0.$$

(3) If $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ is in V then so is cu for any scalar:

$$cx_1+cy_1 = c(x_1+y_1) = c(0) = 0, \qquad cz_1+cw_1 = c(z_1+w_1) = c(0) = 0.$$

b) Not a subspace. Note $u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ are in W, but $u+v$ is not in W.

$$u+v = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad 1\cdot 1 - 0\cdot 0 = 1 \neq 0.$$

3. For (a), answer "YES" if the statement is always true, "NO" if it is always false, and "MAYBE" otherwise.

a) If A is an $m\times n$ matrix and $\mathrm{Nul}(A) = \mathbf{R}^n$, then $\mathrm{Col}(A) = \{0\}$.
YES NO MAYBE

b) Give an example of 2×2 matrix whose column space is the same as its null space.

Solution.

a) If $\mathrm{Nul}(A) = \mathbf{R}^n$ then $Ax = 0$ for all x in $\mathbf{R}^n$, so the only element in $\mathrm{Col}(A)$ is $\{0\}$. Alternatively, the rank theorem says

$$\dim(\mathrm{Col}\ A) + \dim(\mathrm{Nul}\ A) = n \implies \dim(\mathrm{Col}\ A) + n = n \implies \dim(\mathrm{Col}\ A) = 0 \implies \mathrm{Col}\ A = \{0\}.$$

b) Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Its null space and column space are $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

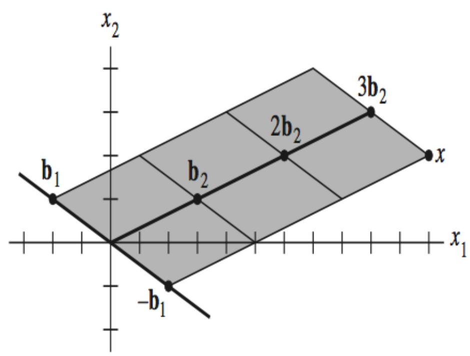
4. Let $\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$, and suppose $[x]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Find x, and draw a picture which clearly represents x as a linear combination of $b_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Solution.

From $[x]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, we have

$$x = -b_1 + 3b_2 = -\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 9 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \end{pmatrix}.$$

The picture below illustrates this.



5. Go back to the 2.8-2.9 worksheet, #3: Find a vector b_3 such that $\{b_1, b_2, b_3\}$ is a basis of $\mathbf{R}^3$.

Solution.

By the increasing span criterion, if we choose b_3 which is not in Span$\{b_1, b_2\}$, then Span$\{b_1, b_2, b_3\}$ will be strictly larger than the (shown in the worksheet) 2-plane V, so it will be a 3-plane within $\mathbf{R}^3$. In other words, the span will be all of $\mathbf{R}^3$.

We could choose $b_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, since the system below is inconsistent.

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 2 & 4 & 0 \\ 2 & 3 & 0 \end{array}\right) \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 3 & -1 \\ 0 & 2 & -1 \end{array}\right) \xrightarrow{R_3 = R_3 - \frac{2}{3}R_2} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -1/3 \end{array}\right).$$