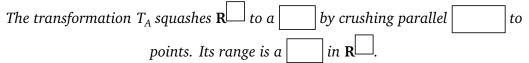
Math 1553 Worksheet §2.8, 2.9

- **1.** Let $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{pmatrix}$ and let T_A be the corresponding linear transformation, defined by $T_A(v) = Av$.
 - a) Find a basis for the nullspace of A and roughly sketch Nul A.
 - **b)** Using the previous part, determine if T_A is one-to-one.
 - c) Find a basis for the column space of *A* and draw a picture of Col *A*.
 - **d)** Using the previous part, determine if T_A is onto.
 - e) Given that we know the nullspace and column space, we can describe the transformation T_A as follows:



Solution.

a) We need to find the parametric vector form for the solutions of Ax = 0. After row reducing, we get the matrix $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. So we have $x_1 = -3x_2 - 2x_3$, $x_2 = x_2$, $x_3 = x_3$ which gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

the nullspace is $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$

The picture is a plane is \mathbf{R}^3 .

So a basis for

- **b)** T_A is not one-to-one, because the nullspace is bigger than just the zero vector.
- c) Only the first column of *A* is a pivot column, so a basis for the columns space is the first column of *A*: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The picture is the line y = 2x in \mathbb{R}^2 .
- **d)** T_A is not onto, since the columns space is only a line in \mathbf{R}^2 , and not all of \mathbf{R}^2 .
- e) The transformation T_A squashes \mathbf{R}^{3} [domain] to a line [column space] by crushing parallel planes [nullspace] to points. Its range is a line [column space] in \mathbf{R}^{2} [codomain].

- **2.** Answer "YES" if the statement is always true, "NO" if it is always false, and "MAYBE" otherwise.
 - a) If *A* is a 3×100 matrix of rank 2, then dim(Nul*A*) = 97. YES NO MAYBE
 - **b)** If *A* is an $m \times n$ matrix and Ax = 0 has only the trivial solution, then the columns of *A* form a basis for \mathbb{R}^m .
 - YES NO MAYBE

Solution.

- a) No. By the Rank Theorem, rank(A) + dim(NulA) = 100, so dim(NulA) = 98.
- **b)** Maybe. If Ax = 0 has only the trivial solution and m = n, then A is invertible, so the columns of A are linearly independent and span \mathbb{R}^m .

If m > n then the statement is false. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the

trivial solution for Ax = 0, but its column space is a 2-dimensional subspace of \mathbb{R}^3 .

3. Consider the following vectors in \mathbf{R}^3 :

$$b_1 = \begin{pmatrix} 2\\2\\2 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 1\\4\\3 \end{pmatrix} \qquad u = \begin{pmatrix} 1\\10\\7 \end{pmatrix}$$

Let $V = \text{Span}\{b_1, b_2\}.$

- **a)** Explain why $\mathcal{B} = \{b_1, b_2\}$ is a basis for *V*.
- **b)** Determine if *u* is in *V*. If so, find $[u]_{\mathcal{B}}$ (the \mathcal{B} -coordinates of *u*).

Solution.

- **a)** A quick check shows that b_1 and b_2 are linearly independent, and we already know they span *V*, so $\{b_1, b_2\}$ is a basis for *V*.
- **b)** *u* is in *V* if and only if $c_1b_1 + c_2b_2 = u$ for some c_1 and c_2 , in which case $[u]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. We form the augmented matrix $\begin{pmatrix} b_1 & b_2 \mid u \end{pmatrix}$ and solve:

$$\begin{pmatrix} 2 & 1 & | & 1 \\ 2 & 4 & | & 10 \\ 2 & 3 & | & 7 \end{pmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{pmatrix} 2 & 1 & | & 1 \\ 0 & 3 & | & 9 \\ 0 & 2 & | & 6 \end{pmatrix} \xrightarrow{R_3 = R_3 - \frac{2}{3}R_2} \begin{pmatrix} 2 & 1 & | & 1 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{pmatrix}.$$

We found $c_1 = -1$ and $c_2 = 3$. This means $-b_1 + 3b_2 = u$, so u is in Span $\{b_1, b_2\}$ and $[u]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.