Math 1553 Supplement §5.1, 5.2

Supplemental Problems

For those who want additional practice problems after completing the worksheet, here are some extra practice problems.

1. Find a basis $\mathcal{B}$ for the (-1)-eigenspace of $Z = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

Solution.

For $\lambda = -1$, we find $\mathrm{Nul}(Z - \lambda I)$.

$$\big(Z - \lambda I \mid 0\big) = \big(Z + I \mid 0\big) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $x = -y$, $y = y$, and $z = 0$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

A basis is $\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. We can check to ensure $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue -1:

$$Z \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2+3 \\ -3+2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

2. Suppose A is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of A. Justify your answer.

Solution.

If λ is an eigenvalue of A and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2 v = \lambda(Av) \implies 0 = \lambda(\lambda v) \implies 0 = \lambda^2 v.$$

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of A.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of A. Therefore, the only eigenvalue of A is 0.

3. Give an example of matrices A and B which have the same eigenvalues and the same algebraic multiplicities for each eigenvalue, but which are *not* similar. Justify why they are not similar.

Solution.

Many examples possible. For example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Both A and B have characteristic equation $\lambda^2 = 0$, so each has eigenvalue $\lambda = 0$ with algebraic multiplicity two. However, the only matrix similar to A is the zero matrix: if P is any invertible 2×2 matrix then $P^{-1}AP = P^{-1}0P = 0$. Therefore, A and B are not similar.

4. Using facts about determinants, justify the following fact: if A is an $n \times n$ matrix, then A and A^T have the same characteristic polynomial.

Solution.

We will use three facts which apply to all $n \times n$ matrices $B,\ Y,\ Z$:

(1) $\det(B) = \det(B^T)$.

(2) $(Y - Z)^T = Y^T - Z^T$

(3) If λ is any scalar then $(\lambda I)^T = \lambda I$ since the identity matrix is completely symmetric about its diagonal.

Using these three facts in order, we find

$$\det(A - \lambda I) = \det\big((A - \lambda I)^T\big) = \det\big(A^T - (\lambda I)^T\big) = \det(A^T - \lambda I).$$

5. Play tic-tac-toe for determinants! Instead of X against O, we have 1 against 0. The 1-player wins if the final matrix has nonzero determinant, while the 0-player wins if the determinant is zero. You can change who goes first, and you can also modify the size of the matrix.

Click the link here, or copy and paste the url below:

`http://people.math.gatech.edu/~jrabinoff6/1718F-1553/demos/tictactoe/tictactoe.html`