Math 1553 Worksheet §5.3 (and some more practice with §5.2)

1. Answer yes / no / maybe. In each case, A is a matrix whose entries are real.

a) If A is a 3×3 matrix with characteristic polynomial $-\lambda(\lambda-5)^2$, then the 5-eigenspace is 2-dimensional.

b) If A is an invertible 2×2 matrix, then A is diagonalizable.

c) If A and B are 3×3 matrices and both have eigenvalues $-1, 0, 1$, then A is similar to B.

d) Suppose A is a 7×7 matrix with four distinct eigenvalues. If one eigenspace has dimension 2, while another eigenspace has dimension 3, then A must be diagonalizable.

Solution.

a) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5- eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(\lambda-5)^2$.

b) Maybe. The identity matrix is invertible and diagonalizable, but the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

c) Yes. In this case, A and B are 3×3 matrices with 3 distinct eigenvalues and thus automatically diagonalizable, and each is similar to $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since A and D are similar, and B and D are similar, it follows that A and B are similar.

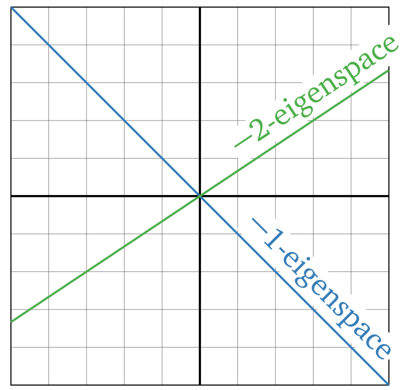
$$A = PDP^{-1} \quad B = QDQ^{-1} \qquad A = PDP^{-1} = PQ^{-1}BQP^{-1} = PQ^{-1}B(PQ^{-1})^{-1}.$$

d) Yes. It is a general fact that every eigenvalue of a matrix has a corresponding eigenspace which is at least 1-dimensional. Given this and the fact that A has four total eigenvalues, we see the sum of dimensions of the eigenspaces of A is at least $2+3+1+1=7$, and in fact must equal 7 since that is the max possible for a 7×7 matrix. Therefore, A has 7 linearly independent eigenvectors and is therefore diagonalizable.

2. Consider the matrix

$$A = -\frac{1}{5}\begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}.$$

a) Find, draw, and label the eigenspaces of A.



We solve:

$$0 = \det(A - \lambda I) = \left(-\frac{8}{5} - \lambda\right)\left(-\frac{7}{5} - \lambda\right) - \left(-\frac{2}{3}\right)\left(-\frac{3}{5}\right) = \frac{56}{25} + 3\lambda + \lambda^2 - \frac{6}{25}$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1), \qquad \text{so } \lambda = -2, \quad \lambda = -1.$$

$$\left(A + 2I \mid 0\right) = \left(\begin{array}{cc|c} \frac{2}{5} & -\frac{3}{5} & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right); \text{ (−2)-eigensp. has basis } \left\{\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}\right\}.$$

$$\left(A + I \mid 0\right) = \left(\begin{array}{cc|c} -\frac{3}{5} & -\frac{3}{5} & 0 \\ -\frac{2}{5} & -\frac{2}{5} & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right); \text{ (−1)-eigensp. has basis } \left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}.$$

b) Compute A^{100}.

From our work in (a), we have $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

We compute $P^{-1} = \dfrac{1}{-5/2}\begin{pmatrix} -1 & -1 \\ -1 & 3/2 \end{pmatrix} = \dfrac{1}{5}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}.$

$$A^{100} = PD^{100}P^{-1} = \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \cdot D^{100}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \cdot 2^{100} & 2 \cdot 2^{100} \\ 2 & -3 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 3 \cdot 2^{100} + 2 & 3 \cdot 2^{100} - 3 \\ 2^{101} - 2 & 2^{101} + 3 \end{pmatrix}.$$