MATH 1553, JANKOWSKI
MIDTERM 1, SPRING 2018, LECTURE C

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Write your section number here: ____________

Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA matches your exam.
- The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work. If you cannot fit your work on the front side of the page, use the back side of the page as indicated.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1.

[2 points for each part]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to justify your answer.

a) **T** **F** The augmented matrix $\left(\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right)$ is in reduced row echelon form.

b) **T** **F** A system of three linear equations in four variables can have exactly one solution.

c) **T** **F** The equation $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 3 \end{pmatrix} x = b$ is consistent for every b in $\mathbf{R}^2$.

d) **T** **F** If A is an $m \times n$ matrix and $Ax = b$ has a unique solution for some b in $\mathbf{R}^m$, then $Ax = 0$ has only the trivial solution.

e) **T** **F** If A is a 3×4 matrix and the solution set for $Ax = 0$ is a line, then A has 2 pivots.

Solution.

a) True.

b) False. If the system is consistent, it will have at least one free variable (4 columns but max of 3 pivots), so it will have infinitely many solutions.

c) True, since $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 4 & 3 \end{pmatrix}$ has a pivot in every row.

d) True. Since $Ax = b$ is consistent, its solution set is a translation of the solution set to $Ax = 0$. Since $Ax = b$ has a unique solution, this means $Ax = 0$ has a unique solution (namely the trivial solution).

e) False. If the solution set to $Ax = 0$ is a line, then we must have exactly one free variable in the homogeneous solution, which means that exactly 3 out of the 4 columns of A will have pivots.

Extra space for scratch work on problem 1

Problem 2. [11 points]

Show your work on parts (a) and (d) (no work necessary for (b) or (c)).

a) Compute $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$.

b) Write three different vectors v_1, v_2, v_3 in $\mathbf{R}^3$ so that $\mathrm{Span}\{v_1, v_2, v_3\}$ is only a plane.

c) Write an *augmented* 3×3 matrix in reduced row echelon form whose corresponding system of linear equations is *inconsistent*, and which has a pivot in every row.

d) Find all solutions to the vector equation

$$x_1 \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 14 \\ -7 \end{pmatrix}.$$

If there are no solutions, justify why the vector equation is inconsistent.

Solution.

a) $3\begin{pmatrix} 2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$.

b) Many possibilities. For example, $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

c) $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$. Also accepted answers like $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ for those who interpreted the wording to mean that we should have a 3×3 matrix to the left of the augment.

d)

$$\left(\begin{array}{cc|c} 2 & 4 & 2 \\ -4 & 1 & 14 \\ 1 & -2 & -7 \end{array}\right) \xrightarrow{R_1 = R_1/2} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ -4 & 1 & 14 \\ 1 & -2 & -7 \end{array}\right) \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 + 4R_1} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 9 & 18 \\ 0 & -4 & -8 \end{array}\right) \xrightarrow[R_3 = R_3/-4]{R_2 = R_2/9} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow[R_1 = R_1 - 2R_2]{R_3 = R_3 - R_2} \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore, $x_1 = -3$ and $x_2 = 2$.

Extra space for work on problem 2

Problem 3. [10 points]

Fairway Frank is infatuated with the system of linear equations given by

$$3x - 4y = 2$$
$$6x + hy = k,$$

where h and k are some real numbers.

a) Determine all values of h and k (if there are any) so that the system of equations is inconsistent.

b) Determine all values of h and k (if there are any) so that the system of equations has infinitely many solutions.

Solution.

$$\left(\begin{array}{cc|c} 3 & -4 & 2 \\ 6 & h & k \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 3 & -4 & 2 \\ 0 & h+8 & k-4 \end{array}\right).$$

a) The system is inconsistent if and only if the right column is a pivot column. Therefore, we need $h+8=0$ and $k-4 \neq 0$ so that the rightmost entry of the second row is a pivot.

$$\boxed{h = -8 \qquad k \neq 4}.$$

b) To have infinitely many solutions, we must have a free variable AND the rightmost column cannot be a pivot colum, so the entire second row above must be zero. Thus $h+8=0$ and $k-4=0$.

$$\boxed{h = -8 \qquad k = 4}.$$

*** In both parts, we are finding h and k so that the lines given by each equation are parallel. In (a), we solve so that we get two *different* parallel lines. In (b), each equation gives the *same* line since the second row is two times the first row.

Extra space for work on problem 3

Problem 4. [12 points]

Consider the system of equations in x_1, x_2, x_3, and x_4 given below.

$$\begin{aligned} x_1 - x_2 - 2x_3 + 2x_4 &= -7 \\ -x_1 + x_2 + x_3 - 2x_4 &= 5 \\ -4x_1 + 4x_2 + 6x_3 - 8x_4 &= 24. \end{aligned}$$

a) Write this system of linear equations as a vector equation.

b) Write this system of linear equations as a matrix equation $Ax = b$. Specify every entry of A, x, and b.

c) Put an augmented matrix into reduced row echelon form to solve the system of equations. Write your answer in parametric vector form.

Solution.

a) $$x_1\begin{pmatrix}1\\-1\\-4\end{pmatrix} + x_2\begin{pmatrix}-1\\1\\4\end{pmatrix} + x_3\begin{pmatrix}-2\\1\\6\end{pmatrix} + x_4\begin{pmatrix}2\\-2\\-8\end{pmatrix} = \begin{pmatrix}-7\\5\\24\end{pmatrix}.$$

b) $$\begin{pmatrix}1 & -1 & -2 & 2\\-1 & 1 & 1 & -2\\-4 & 4 & 6 & -8\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-7\\5\\24\end{pmatrix}.$$

c)

$$\left(\begin{array}{cccc|c}1 & -1 & -2 & 2 & -7\\-1 & 1 & 1 & -2 & 5\\-4 & 4 & 6 & -8 & 24\end{array}\right) \xrightarrow[R_3=R_3+R_1]{R_2=R_2+R_1} \left(\begin{array}{cccc|c}1 & -1 & -2 & 2 & -7\\0 & 0 & -1 & 0 & -2\\0 & 0 & -2 & 0 & -4\end{array}\right) \xrightarrow[R_3/-2]{R_2=R_2\cdot(-1)} \left(\begin{array}{cccc|c}1 & -1 & -2 & 2 & -7\\0 & 0 & 1 & 0 & 2\\0 & 0 & 1 & 0 & 2\end{array}\right) \xrightarrow[R_1=R_1+2R_2]{R_3=R_3-R_2} \left(\begin{array}{cccc|c}1 & -1 & 0 & 2 & -3\\0 & 0 & 1 & 0 & 2\\0 & 0 & 0 & 0 & 0\end{array}\right)$$

The free variables are x_2 and x_4: $x_1 = x_2 - 2x_4 - 3$, $x_2 = x_2$, $x_3 = 2$, $x_4 = x_4$.

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}x_2 - 2x_4 - 3\\x_2\\2\\x_4\end{pmatrix} = \begin{pmatrix}x_2\\x_2\\0\\0\end{pmatrix} + \begin{pmatrix}-2x_4\\0\\0\\x_4\end{pmatrix} + \begin{pmatrix}-3\\0\\2\\0\end{pmatrix} = \boxed{x_2\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}-2\\0\\0\\1\end{pmatrix} + \begin{pmatrix}-3\\0\\2\\0\end{pmatrix}}.$$

Extra space for work on problem 4

Problem 5. [7 points]

Parts (a) and (b) are unrelated.

a) Write a 3×3 matrix A in reduced row echelon form, with the property that the solution set to $Ax = 0$ is $\text{Span}\left\{\begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$. Briefly justify your answer.

b) Write a vector b in $\mathbf{R}^3$ which is *not* a linear combination of $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\-1\end{pmatrix}$.

You do not need to justify your answer.

Solution.

a) Any solution $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ to $Ax = 0$ is a scalar multiple of $\begin{pmatrix}1\\0\\2\end{pmatrix}$, so $x_1 = \dfrac{x_3}{2}$ and $x_2 = 0$.

The equations $x_1 - \dfrac{x_3}{2} = 0$ and $x_2 = 0$ are just $\left(1 \quad 0 \quad -\tfrac{1}{2} \mid 0\right)$ and $\left(0 \quad 1 \quad 0 \mid 0\right)$. Therefore, the first two rows of A are $\left(1 \quad 0 \quad -\tfrac{1}{2}\right)$ and $\left(0 \quad 1 \quad 0\right)$. To fill the remaining row of A and make sure that A is in RREF, we just put a row of zeros.

$$A = \begin{pmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

We can check: $Ax = 0$ precisely when $x_1 - \frac{x_3}{2} = 0$ and $x_2 = 0$, which is exactly what we wanted.

b) Just like in the practice exam, we see that any vector in the span of $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\-1\end{pmatrix}$ must have its first entry equal to its second entry, so a vector like $\begin{pmatrix}1\\0\\0\end{pmatrix}$ cannot be in the span.

Alternatively, you can use numbers for any inconsistent system corresponding to $\left(\begin{array}{cc|c}1 & 0 & a\\ 1 & 0 & b\\ 0 & -1 & c\end{array}\right)$. If we choose $a \neq b$ then the first step of row reduction will give $\left(\begin{array}{cc|c}1 & 0 & a\\ 0 & 0 & b-a\\ 0 & -1 & c\end{array}\right)$ which puts a pivot in the right column if $a \neq b$.

Extra space for work on problem 5