MATH 1553, SPRING 2018
SAMPLE MIDTERM 2 (VERSION A), 1.7 THROUGH 2.9

Name		**Section**	

Please **read all instructions** carefully before beginning.

- You have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Please show your work unless specified otherwise. A correct answer without appropriate work may be given little or no credit.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §1.7 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§1.7 through 2.9.

Problem 1.

[Parts a) through f) are worth 2 points each]

a) Complete the following definition (be mathematically precise!):
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is *linearly independent* if...

b) Let $A = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$. If A is invertible, find A^{-1}. If A is not invertible, justify why.

The remaining problems are true or false. Answer true if the statement is *always* true. Otherwise, answer false. You do not need to justify your answer.

c) **T** **F** If A is an $n \times n$ matrix and the columns of A span $\mathbf{R}^n$, then $Ax = 0$ has only the trivial solution.

d) **T** **F** If A is a 6×7 matrix and the null space of A has dimension 4, then the column space of A is a 2-plane.

e) **T** **F** If A is an $n \times n$ matrix and $Ax = b$ has exactly one solution for some b in $\mathbf{R}^n$, then A is invertible.

f) **T** **F** If A is an $m \times n$ matrix and $m > n$, then the linear transformation $T(x) = Ax$ cannot be one-to-one.

Solution.

a) the equation $x_1 v_1 + \cdots + x_p v_p = 0$ has only the trivial solution $x_1 = \cdots = x_p = 0$.

b) $\det(A) = 2 - (-2) = 4$, so A is invertible; $A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

c) True. A is invertible by the Invertible Matrix Theorem, so $Ax = 0$ has only the trivial solution.

d) False. By the Rank Theorem, $\dim(\text{Col } A) + \dim(\text{Nul } A) = 7$, so $\dim(\text{Col } A) = 3$.

e) True. If $Ax = b$ has exactly one solution for some b, then $Ax = 0$ has exactly one solution (since the sol. set for $Ax = b$ is a translate of the sol. set for $Ax = 0$), hence A is invertible.

f) False. $T : \mathbf{R}^n \to \mathbf{R}^m$ can be one to one. For example, $T(a) = (a, 0)$.

Problem 2.

[10 points]

Parts (a), (b), and (c) are unrelated.

a) Let $V = \text{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

Fill in the blank: the dimension of V is ________.

b) Let $W = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - y - z = 0\right\}$.

Is W a subspace of $\mathbf{R}^3$? (no justification required)

YES NO

c) The famous philologist is obsessed with the set of vectors

$$\left\{\begin{pmatrix}-1\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\h\\-7\end{pmatrix}\right\}$$

where h is some real number.
Find all values of h that make the set linearly dependent.

Solution.

a) A basis for V is $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$, so $\dim V = 2$.

b) W is a subspace of $\mathbf{R}^3$. In fact, it is Nul A for the matrix $A = \begin{pmatrix}1 & -1 & -1\end{pmatrix}$.

c)

$$\begin{pmatrix}-1 & 1 & 1\\ 3 & 1 & h\\ -1 & -1 & -7\end{pmatrix} \xrightarrow[R_3=R_3-R_1]{R_2=R_2+3R_1} \begin{pmatrix}-1 & 1 & 1\\ 0 & 4 & h+3\\ 0 & -2 & -8\end{pmatrix} \xrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix}-1 & 1 & 1\\ 0 & -2 & -8\\ 0 & 4 & h+3\end{pmatrix} \xrightarrow{R_3=R_3+2R_2} \begin{pmatrix}-1 & 1 & 1\\ 0 & -2 & -8\\ 0 & 0 & h-13\end{pmatrix}.$$

The vectors are linearly dependent if and only if the matrix has fewer than 3 pivots. The matrix will have three pivots unless $h - 13 = 0$, which is when $\boxed{h = 13}$.

Problem 3.

[11 points]

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation given by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 - x_2 \\ 2x_1 \end{pmatrix},$$

and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be reflection about the line $y = x$.

a) Write the standard matrix A for T and the standard matrix B for U.

b) Is U one-to-one? Briefly justify your answer.

c) Find the standard matrix for $U \circ T$.

d) Is the transformation $U \circ T$ onto? Briefly justify your answer.

Solution.

a) $A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix}$, and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

b) U is one-to-one, since B has a pivot in every column. Alternatively, if $U(x, y)$ is the zero vector then $(y, x) = (0, 0)$ so $x = y = 0$, which shows U is one-to-one.

c) The matrix for $U \circ T$ is

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

d) $U \circ T$ is onto, since BA has a pivot in every row.

Problem 4. [10 points]

Consider the following matrix A and its reduced row echelon form:

$$\begin{pmatrix} 1 & -2 & 4 \\ 0 & 0 & 1 \\ 1 & -2 & 3 \\ -2 & 4 & -8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) Find a basis for Nul A.

b) Find a basis $\mathcal{B}$ for Col A.

c) Let $x = \begin{pmatrix} -2 \\ -1 \\ -1 \\ 4 \end{pmatrix}$. Is x in Col A?

If your answer is no, justify why x is not in Col A.
If your answer is yes, find $[x]_{\mathcal{B}}$.

Solution.

a) From the RREF of $\left(A \mid 0\right)$ we see that $Ax = 0$ when $x_1 = 2x_2$, $x_2 = x_2$, and $x_3 = 0$. In parametric vector form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \text{so a basis for Nul } A \text{ is } \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

b) The RREF of A shows that the first and third columns are pivot columns.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 3 \\ -8 \end{pmatrix} \right\}.$$

c) We attempt to solve $x = c_1 b_1 + c_2 b_2$ for some scalars c_1 and c_2.

$$\left(\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & 1 & -1 \\ 1 & 3 & -1 \\ -2 & -8 & 4 \end{array}\right) \xrightarrow[R_4 = R_4 + 2R_1]{R_3 = R_3 - R_1} \left(\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1 = R_1 - 4R_2]{R_3 = R_3 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

So $c_1 = 2$ and $c_2 = -1$, and $x = 2b_1 - b_2$.

$$[x]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Problem 5. [7 points]

Parts (a) and (b) are unrelated.

a) Suppose that a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ satisfies $T\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Find $T\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

b) Write a single matrix A that satisfies both of the following two properties:

- Col A is a subspace of $\mathbf{R}^4$, and
- Nul A is the line $y = 10x$ in $\mathbf{R}^2$.

Solution.

a) $\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, so by linearity of T,

$$T\begin{pmatrix} 4 \\ -1 \end{pmatrix} = T\begin{pmatrix} 1 \\ -2 \end{pmatrix} + T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

b) Our A must be 4×2, so that Col A is a subspace $\mathbf{R}^4$ and Nul A is a subspace of $\mathbf{R}^2$. The line $y = 10x$ is spanned by $\begin{pmatrix} 1 \\ 10 \end{pmatrix}$. Therefore, if $\begin{pmatrix} a & b \end{pmatrix}$ is a row of A, then

$$0 = \begin{pmatrix} a & b \end{pmatrix}\begin{pmatrix} 1 \\ 10 \end{pmatrix} = a + 10b.$$

Thus, $a = -10b$ and $b = b$, so a row of A is $\begin{pmatrix} -10 & 1 \end{pmatrix}$ or any scalar multiple of it. We need exactly one free variable since Nul A is a line.

$$A = \begin{pmatrix} -10 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$ is one example.

[Scratch work]