MATH 1553, SPRING 2018
SAMPLE MIDTERM 3 (VERSION A), 3.1-5.5

Name		**GT Email**	@gatech.edu

Write your section number here: ____________

Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA matches your exam.
- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to chapter 3 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to chapters 3 and 5.

Problem 1.

[Parts a) through f) are worth 2 points each]

a) Suppose A is a 3×3 matrix whose entries are real numbers. How many *distinct* real eigenvalues can A possibly have? Circle all that apply.

(a) 0

(b) 1

(c) 2

(d) 3

The remaining problems are true or false. Answer true if the statement is *always* true. Otherwise, answer false. You do not need to justify your answer. In every case, assume that the entries of the matrix A are real numbers.

b) **T** **F** If A is an $n \times n$ matrix then $\det(-A) = -\det(A)$.

c) **T** **F** If v is an eigenvector of a square matrix A, then $-v$ is also an eigenvector of A.

d) **T** **F** If A is an $n \times n$ matrix and $\lambda = 2$ is an eigenvalue of A, then $\text{Nul}(A - 2I) = \{0\}$.

e) **T** **F** If A is a 3×3 matrix with characteristic polynomial $(3-\lambda)^2(2-\lambda)$, then the eigenvalue $\lambda = 2$ must have geometric multiplicity 1.

f) **T** **F** The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

Solution.

a) Circle (b), (c), and (d). There must be at least one real eigenvalue since n is odd, and there can be two (e.g. $(1-\lambda)^2(5-\lambda)$) or even three.

b) False. Since $\det(cA) = c^n \det(A)$, we see $\det(-A) = (-1)^n \det(A) = \det(A)$ if n is even.

c) True. Straight from the chapter 5 homework. If v is an eigenvector then so is cv for any $c \neq 0$.

d) False. $\mathrm{Nul}(A - 2I)$ is the 2-eigenspace, which is never just the zero vector if 2 is an eigenvalue.

e) True. The geometric multiplicity of an eigenvalue is always at least 1 but never more than the algebraic multiplicity (which here is 1).

f) True. Every 2×2 matrix with eigenvalues $\lambda = 2$ and $\lambda = 3$ is similar to $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and to $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ depending upon how you place the eigenvectors, so the matrices are similar. Or you could observe $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = P \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}$ where $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Extra space for scratch work on problem 1

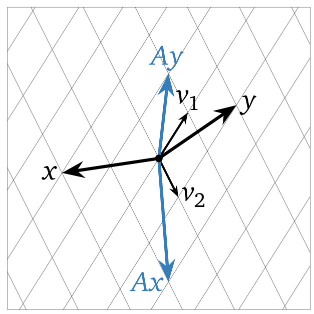
Problem 2.

[10 points]

Short answer. For (a) and (b), show any brief computations. For (c), (d), and (e), you do not need to justify your answer. In each case, assume the entries of A and B are real numbers.

a) Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 7 \end{pmatrix}$, and define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = Ax$. Find the area of $T(S)$, if S is a triangle in $\mathbf{R}^2$ with area 2.

b) Suppose that $A = P \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix} P^{-1}$, where P has columns v_1 and v_2. Given x and y in the picture below, draw the vectors Ax and Ay.



c) Write a 2×2 matrix A which is not diagonalizable and not invertible.

d) Give an example of 2×2 matrices A and B which have the same characteristic polynomial but are not similar.

e) Write a diagonalizable 3×3 matrix A whose only eigenvalue is $\lambda = 2$.

Solution.

a) $|\det(A)|\mathrm{Vol}(S) = |-7-1| \cdot 2 = 16.$

b) A does the same thing as $D = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}$, but in the v_1, v_2-coordinate system. Since D scales the first coordinate by $1/2$ and the second coordinate by -1, hence A scales the v_1-coordinate by $1/2$ and the v_2-coordinate by -1.

c) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

d) For example, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

e) There is only one such matrix: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Extra space for work on problem 2

Problem 3. [10 points]

Let $A = \begin{pmatrix} 2 & -4 \\ 1 & 2 \end{pmatrix}$.

a) Find the eigenvalues of A.

b) Let λ be the eigenvalue of A whose imaginary part is negative. Find an eigenvector of A corresponding to λ.

c) Find a matrix C which is similar to A and represents a composition of scaling and rotation.

d) What is the scaling factor for C?

e) Find the angle of rotation for C.
(do not leave your answer in terms of arctan; the answer is a standard angle).

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} 2-\lambda & -4 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2+4 = \lambda^2-4\lambda+8, \qquad \lambda = \frac{4 \pm \sqrt{4^2-32}}{2} = \frac{4\pm\sqrt{-16}}{2} = 2\pm 2i.$$

b) $\left(A-(2-2i)I \quad 0\right) = \begin{pmatrix} 2i & -4 & 0 \\ 1 & 2i & 0 \end{pmatrix} = \left(\begin{array}{cc|c} 2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right)$. One eigenvector is $v = \begin{pmatrix} -4 \\ -2i \end{pmatrix}$, or $\begin{pmatrix} 4 \\ 2i \end{pmatrix}$. Alternatively, we could row reduce $\left(\begin{array}{cc|c} 2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1=R_1/(2i)} \begin{pmatrix} 1 & 2i & 0 \end{pmatrix}$, so $\begin{pmatrix} -2i \\ 1 \end{pmatrix}$ is an eigenvector. Really, any nonzero multiple of $\begin{pmatrix} -4 \\ -2i \end{pmatrix}$ is an eigenvector.

c) We have C_1 and C_2 as possibilities for C, done as follows.
If we use $\lambda = 2-2i$ then $C_1 = \begin{pmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$.
However, if we use $\lambda = 2+2i$ then $C_2 = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$.

d) $|\lambda| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$ (it is fine if the student leaves it as $\sqrt{8}$).

e) Use the $-\arg(\lambda)$ formula or just use knowledge of rotations (which we do below).

$$C_1 = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} = 2\sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \cos(\theta_1) = \sin(\theta_1) = \frac{1}{\sqrt{2}}, \text{ so } \theta_1 = \frac{\pi}{4}.$$

If the student used C_2, then the angle is different:

$$C_2 = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} = 2\sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad \cos(\theta_2) = \frac{1}{\sqrt{2}}, \ \sin(\theta_2) = -\frac{1}{\sqrt{2}}, \text{ so } \theta_2 = -\frac{\pi}{4}.$$

Extra space for work on problem 3

Problem 4. [9 points]

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

a) Find the eigenvalues of A, and find a basis for each eigenspace.

b) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix P so that $A = PDP^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} 2-\lambda & 3 & 1 \\ 3 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1)^6 \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (-1-\lambda)((2-\lambda)^2 - 9)$$

$$= (-1-\lambda)(\lambda^2 - 4\lambda - 5) = -(\lambda+1)^2(\lambda - 5).$$

So $\lambda = -1$ and $\lambda = 5$ are the eigenvalues.

$\underline{\lambda = -1}$: $\big(A + I \mid 0\big) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/3]{R_1 = R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$,

with solution $x_1 = -x_2$, $x_2 = x_2$, $x_3 = 0$. The (-1)-eigenspace has basis $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

$\underline{\lambda = 5}$:

$$\big(A - 5I \mid 0\big) = \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \xrightarrow[R_3 = R_3/(-6)]{R_2 = R_2 + R_1} \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\text{then } R_2 \leftrightarrow R_3,\ R_1/(-3)]{R_1 = R_1 - R_3,\ R_2 = R_2 - 5R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

with solution $x_1 = x_2$, $x_2 = x_2$, $x_3 = 0$. The 5-eigenspace has basis $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

b) A is a 3×3 matrix that only admits 2 linearly independent eigenvectors, so A is not diagonalizable.

Extra space for work on problem 4

Problem 5.

[9 points]

Parts (a) and (b) are not related.

a) Suppose we know that

$$\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^{-1}.$$

Find $\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix}^{98}$.

b) Let B be a 4×4 matrix satisfying $\det(B) = 2$, and let

$$C = \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 3 \\ -1 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Find $\det(CB^{-1})$.

Solution.

a)

$$\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix}^{98} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}^{98} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -4 & 10 \\ -2 & 5 \end{pmatrix}.$$

b) We use the cofactor expansion along the second column:

$$\det(C) = 1(-1)^{3+2} \det \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix} = -1 \cdot 2 \cdot (-1)^2 \det \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} = -2 \cdot (-2 - 3) = 10.$$

Therefore,

$$\det(CB^{-1}) = \det(C) \det(B^{-1}) = 10 \cdot \frac{1}{2} = 5.$$

Extra space for work on problem 5