MATH 1553, JANKOWSKI
MIDTERM 3, SPRING 2018, LECTURE A

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Write your section number here: ____________

Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA matches your exam.
- The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Show your work unless specified otherwise. A correct answer without appropriate work will receive little or no credit.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1.

[2 points each]

On problem 1, you do not need to justify your answer, and there is no partial credit.

a) Write a 2×2 matrix A which is invertible but not diagonalizable.

The remaining problems are true or false. Answer true if the statement is *always* true. Otherwise, answer false. In every case, assume that the entries of the matrix A are real numbers.

b) **T** **F** If A is the 3×3 matrix satisfying $Ae_1 = e_2$, $Ae_2 = e_3$, and $Ae_3 = e_1$, then $\det(A) = 1$.

c) **T** **F** If A is an $n \times n$ matrix and $\det(A) = 2$, then 2 is an eigenvalue of A.

d) **T** **F** The matrices $A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix}$ are similar.

e) **T** **F** If A is an $n \times n$ matrix and v and w are eigenvectors of A, then $v + w$ is also an eigenvector of A.

f) **T** **F** If A is an invertible $n \times n$ matrix and B is similar to A, then B is invertible.

Solution.

a) Many answers possible. For example, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

b) True. $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. You can compute $\det(A) = 1$ or just do two row swaps to get the identity matrix, so that $\det(A) = (-1)^2 = 1$.

c) False. For example, $A = \begin{pmatrix} 4 & 0 \\ 0 & 1/2 \end{pmatrix}$ has $\det(A) = 2$ but its eigenvalues are 4 and $\frac{1}{2}$.

d) True. They are 2×2 matrices with the same two distinct eigenvalues, and are similar to each other because they are both similar to $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

e) False. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors, but $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not an eigenvector.

f) True. A and B have the same eigenvalues. Since 0 is not an eigenvalue of A (A is invertible), it is not an eigenvalue of B, so B is invertible.

Extra space for scratch work on problem 1

Problem 2. [8 points]

Short answer. Show your work on part (b). In every case, the entries of each matrix must be real numbers.

a) Write a 2×2 matrix A for which $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are eigenvectors corresponding to the same eigenvalue.

b) Find the area of the triangle with vertices $(0,0)$, $(1,4)$, and $(4,2)$.

c) Write a 3×3 matrix A with only one real eigenvalue $\lambda = 4$, such that the 4-eigenspace for A is a two-dimensional plane in $\mathbf{R}^3$.

d) Suppose A is an $n\times n$ matrix. Which of the following must be true? Circle all that apply.

I. If $\det(A) = 0$ then A is not invertible.

II. If A is diagonalizable, then A has n distinct eigenvalues.

Solution.

a) Any scalar multiple of the identity will work, for example $A = \begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$.

b) The area is $\frac{1}{2}\left|\det\begin{pmatrix}1 & 4\\4 & 2\end{pmatrix}\right| = \frac{1}{2}|2-16| = 7$.

c) Many examples possible. For example, $A = \begin{pmatrix}4 & 0 & 0\\0 & 4 & 1\\0 & 0 & 4\end{pmatrix}$.

d) (I) is correct.

Extra space for work on problem 2

Problem 3. [10 points]

Consider the matrix

$$A = \begin{pmatrix} 3 & -7 \\ 1 & -1 \end{pmatrix}$$

a) Find all eigenvalues of A. Simplify your answer.

b) For the eigenvalue with negative imaginary part, find an eigenvector.

c) Using the eigenvalue with negative imaginary part, find a matrix C that is a composition of of rotation and scaling and which is similar to A.

d) Write the scale factor for C.

e) By what counterclockwise angle does your matrix C rotate? Simplify your answer (don't leave it in terms of arctan), as it is a standard angle.

Solution.

a) We compute the characteristic equation:

$$0 = \det(A - \lambda I) = (3-\lambda)(-1-\lambda) + 7 = \lambda^2 - 2\lambda - 3 + 7 = \lambda^2 - 2\lambda + 4.$$

By the quadratic formula,

$$\lambda = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm \sqrt{3}i.$$

b) Let $\lambda = 1 - \sqrt{3}i$. Then

$$A - \lambda I_2 = \begin{pmatrix} 2+\sqrt{3}i & -7 \\ \star & \star \end{pmatrix} \quad \Longrightarrow \quad v = \begin{pmatrix} 7 \\ 2+\sqrt{3}i \end{pmatrix}$$

is an eigenvector for λ.

c) Using the eigenvalue $\lambda = 1 - \sqrt{3}i$ we get

$$C = \begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix} = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

d) C scales by a factor of $\sqrt{1^2 + (-\sqrt{3})^2} = 2$.

e) $\overline{\lambda} = 1 + \sqrt{3}i$, so $\arg(\overline{\lambda})$ is in the first quadrant with tangent equal to $\sqrt{3}$, so $\theta = \frac{\pi}{3}$.

Alternatively, we could pull out the scale factor and find $\cos(\theta) = \frac{1}{2}$ and $\sin(\theta) = \frac{\sqrt{3}}{2}$, so $\theta = \frac{\pi}{3}$.

Extra space for work on problem 3

Problem 4. [10 points]

Let $A = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{pmatrix}$.

a) Find the eigenvalues of A.

b) Find a basis for each eigenspace of A. Mark your answers clearly.

c) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix P so that $A = PDP^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} -1-\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 4-\lambda \end{pmatrix} = (2-\lambda)(-1)^4 \det\begin{pmatrix} -1-\lambda & -2 \\ 3 & 4-\lambda \end{pmatrix}$$
$$= (2-\lambda)((-1-\lambda)(4-\lambda)+6) = (2-\lambda)(\lambda^2-3\lambda-4+6)$$
$$= (2-\lambda)(\lambda^2-3\lambda+2) = (2-\lambda)(\lambda-2)(\lambda-1)$$

So $\lambda = 1$ and $\lambda = 2$ are the eigenvalues.

$\underline{\lambda = 1}$: $\left(A - I \mid 0\right) = \left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 3 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = -R_1/2]{R_3 = R_3 + \frac{3}{2}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ with solution

$x_1 = -x_3$, $x_2 = 0$, $x_3 = x_3$. The 1-eigenspace has basis $\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

$\underline{\lambda = 2}$:

$$\left(A - 2I \mid 0\right) = \left(\begin{array}{ccc|c} -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = -R_1/3]{R_3 = R_3 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

with solution $x_1 = -\frac{2}{3}x_3$, $x_2 = x_2$, $x_3 = x_3$. The 2-eigenspace has basis $\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}\right\}$.

b) A is diagonalizable; $A = PDP^{-1}$ where $P = \begin{pmatrix} -1 & 0 & -2/3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Extra space for work on problem 4

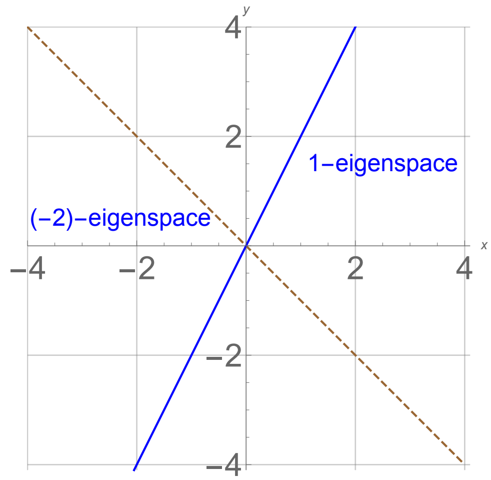
Problem 5.

[10 points]

Parts (a) and (b) are not related.

a) Find $\det(A^3)$ if $A = \begin{pmatrix} 1 & -3 & 4 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 0 & -1 & 20 \end{pmatrix}$.

b) Find the 2×2 matrix A whose eigenspaces are drawn below. Fully simplify your answer. (to be clear: the dashed line is the (-2)-eigenspace).



Solution.

a) Using the cofactor expansion along the second row, we find

$$\det(A) = -2(-1)^5 \det\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 3 \\ 2 & 0 & 20 \end{pmatrix} = 2\big(20 + 3(-6) + 2(-2)\big) = 2(20-18-4) = -4,$$

so $\det(A^3) = (-4)^3 = -64$.

b) From the picture, we see $\lambda_1 = 1$ has eigenvector $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Also, $\lambda = -2$ has eigenvector $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Forming $P = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ we get $A = PDP^{-1}$, so

$$\begin{aligned} A &= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} -3 & 3 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}. \end{aligned}$$

Extra space for work on problem 5