Instructions: Complete 5 of the 6 problems, and circle their numbers below – the uncircled problems will not be graded.

1. Let $G$ be a 2-connected graph with at least 4 vertices. Show that there exists an edge $e$ such that $G/e$, the graph obtained from $G$ by contracting $e$, is still 2-connected.

2. Let $G$ be a graph and $d$ a positive integer. Suppose $e(H) < d|H|$ for all subgraphs $H$ of $G$. Show that $G$ is $(2d)$-colorable.

3. Let $G$ be a connected, triangle-free graph and assume that no four vertices of $G$ induce a subgraph whose edge set is a matching of size 2 in $G$. Prove that $G$ is 3-colorable. (Hint: Consider an induced odd cycle in $G$.)

4. Show that there exists a 2-coloring of the edges of the complete $n$-vertex graph $K_n$ that contains at most $\frac{1}{4}\binom{n}{3}$ many monochromatic triangles.

5. The diameter $\text{diam}(G)$ of a graph $G$ is the maximum distance between two vertices of $G$, where distance is the length of shortest path (so only the complete graph has diameter one). Let $G(n,p)$ denote the binomial random graph with vertex set $[n]$, i.e., where each of the $\binom{n}{2}$ possible edges is inserted independently with probability $p$. For constant edge-probability $p \in (0, 1)$, show that $\mathbb{P}(\text{diam}(G(n,p)) = 2) \to 1$ as $n \to \infty$.

6. As before, by $G(n,p)$ we denote the binomial random graph with vertex set $[n]$. Let $\omega = \omega(n) \to \infty$ as $n \to \infty$. For any $p = p(n) \in [0,1]$ show that

$$\mathbb{P}(G(n,p) \text{ is a forest}) = \begin{cases} 1 - o(1) & \text{if } p \leq \omega^{-1} n^{-1}, \\ o(1) & \text{if } p \geq \omega n^{-1}, \end{cases}$$

providing full/complete details for your reasoning.

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