Analysis Comprehensive Exam
Fall 2017

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let $\nu$ be a signed Borel measure on $\mathbb{R}$. Prove that

$$\nu(E) = \sup \left\{ \left| \int_E f \, d\nu \right| : |f| \leq 1 \right\}.$$ 

2. Assume that $1 \leq p < q \leq \infty$ and $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Show that $f \in L^r(\mathbb{R}^d)$ for all $p \leq r \leq q$.

3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is monotone increasing, and that we have $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 1$. Prove that $f$ is absolutely continuous on every finite interval $[a, b]$ if and only if

$$\int_{-\infty}^{\infty} f'(x) \, dx = 1.$$ 

4. Let $\phi$ be a function in $C^1(\mathbb{R}^d)$ with compact support. Show that for any $f \in L^1(\mathbb{R}^d)$ the function

$$\phi \ast f(x) := \int \phi(x - y) f(y) \, dy$$

is differentiable.

5. The two parts of this problem are not related.

(a) Exhibit a set $E \subseteq [0, 1]$ that is meager yet has measure $|E| = 1$.

(b) Suppose that $f$ is an infinitely differentiable function on $\mathbb{R}$ such that for each $x \in \mathbb{R}$ there exists some integer $n_x \geq 0$ so that $f^{(n_x)}(x) = 0$. Prove that there exists some open interval $(a, b)$ and some polynomial $p$ such that $f(x) = p(x)$ for all $x \in (a, b)$.

6. Minimize

$$\int_{\mathbb{R}} x^2 f(x) \, dx$$

among all measurable functions with $0 \leq f(x) \leq A$ and $\int_{\mathbb{R}} f(x) \, dx = B$.

7. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a Banach space $X$. Fix $1 \leq p \leq \infty$ and let $p'$ the dual index to $p$. Given a linear functional $\mu$ in the dual space $X^*$, set $T(\mu) = (\mu(x_n))_{n \in \mathbb{N}}$, and suppose that $T(\mu) \in \ell^{p'}$ for every $\mu \in X^*$. Prove that $T$ is a bounded mapping of $X^*$ into $\ell^{p'}$. 

Turn Over \rightarrow.
8. Let \( f \in L^1(\mathbb{R}^2) \) be a function such that \( f(R^{-1}x) = f(x) \) for all \( x \in \mathbb{R}^2 \), where

\[
R = \begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}.
\]

Show that if \( \alpha \) is an irrational multiple of \( 2\pi \) then for a.e. \( x \in \mathbb{R}^2 \),

\[
f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(R_\theta x) \, d\theta
\]

where

\[
R_\theta = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]