Analysis Comprehensive Exam
Spring 2021

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.

NOTES:
- All functions in this exam are real-valued unless specified otherwise.
- The exterior Lebesgue measure of $E \subseteq \mathbb{R}^d$ is denoted by $|E|_e$, and if $E$ is measurable then its Lebesgue measure is $|E|$. 
1. Let \( g_n : [0, 1] \to \mathbb{R} \) be measurable for \( n \geq 1 \). Also let \( g : [0, 1] \to \mathbb{R} \). Assume that \( \{g_n\} \) increase to \( g \), that is for each \( t \in [0, 1] \),
\[
g_1(t) \leq g_2(t) \leq g_3(t) \leq \cdots
\]
and
\[
\lim_{n \to \infty} g_n(t) = g(t).
\]
Let
\[
g^+(t) = \max \{0, g(t)\}, \quad t \in [0, 1]
\]
and assume that \( g^+ \) is Lebesgue integrable on \([0, 1]\). Evaluate
\[
\lim_{n \to \infty} \frac{1}{n} \int_0^1 \ln \left( 1 + e^{ng_n(t)} \right) dt
\]
and justify your limit.

**Definitions.** (a) A linear mapping \( A : H \to H \) is a topological isomorphism if it is a bijection and both \( A \) and \( A^{-1} \) are bounded. (b) Two inner products for a Hilbert space \( H \) are equivalent if the two norms induced from those inner products are equivalent norms for \( H \).

2. Let \( H \) be a Hilbert space, and assume that \( A : H \to H \) is bounded, linear, and positive definite. Prove the following statements.

(a) \( A \) is injective and has dense range.

(b) \( A \) is a topological isomorphism that maps \( H \) onto itself if and only if \( A \) is surjective.

(c) If \( A \) is surjective and positive definite, then
\[
[x, y] = \langle Ax, y \rangle, \quad \text{for } x, y \in H,
\]
defines an inner product \([\cdot, \cdot]\) on \( H \) that is equivalent to the original inner product \( \langle \cdot, \cdot \rangle \).

3. Let \( f : [a, b] \to \mathbb{R} \cup \{-\infty, \infty\} \) be (Lebesgue) integrable. Let \( \varepsilon > 0 \). Show there exists a closed set \( F \) of \([a, b]\) with \( |[a, b] \setminus F| < \varepsilon \), and a sequence of polynomials \( \{p_n\} \) such that
\[
\sup_{x \in F} |f(x) - p_n(x)| \to 0 \text{ as } n \to \infty.
\]

**Hint:** You may assume that given \( \delta > 0 \), there exists a continuous function \( C : [a, b] \to \mathbb{R} \) with
\[
\int_a^b |f(x) - C(x)| \, dx < \delta.
\]
4. Suppose that $U$ is an unbounded open subset of $\mathbb{R}$.

(a) For each $n \in \mathbb{N}$, set $A_n = \bigcup_{|k| > n, k \in \mathbb{Z}} U/k$, where $U/k = \{x/k : x \in U\}$. Prove that $A_n$ is dense in $\mathbb{R}$.

(b) Prove that the set

$$A = \{x \in \mathbb{R} : kx \in U \text{ for infinitely many } k \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$.

5. Let $(S, \Sigma, \nu)$ be a measure space. Assume that

$$S = \bigcup_{n=1}^{\infty} E_n$$

where the $\{E_n\}_{n=1}^{\infty}$ are disjoint measurable sets, each with $\nu(E_n) < \infty$. Define $\mu$ on $S$ by

$$\mu(B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\nu(B \cap E_n)}{\nu(E_n)}, \quad B \in \Sigma.$$ 

(a) Show that $\mu$ is a finite measure on $S$.

(b) Show that $\mu$ is absolutely continuous on $S$ with respect to $\nu$ and $\nu$ is absolutely continuous on $S$ with respect to $\mu$.

(c) Find a function $f : S \to \mathbb{R}$ such that for all $A \in \Sigma$,

$$\mu(A) = \int_A f \, d\nu.$$ 

6. Assume that $E \subseteq \mathbb{R}^d$ is measurable, with $0 < |E| < \infty$. Set

$$f(t) = |E \cap (E + t)|, \quad \text{for } t \in \mathbb{R}^d,$$

where $E + t = \{x + t : x \in E\}$. Show that $f(t) \to 0$ as $\|t\| \to \infty$.

7. (a) Let $E \subseteq \mathbb{R}^n$. Let $f : E \to [0, \infty)$ be measurable. Define its distribution function

$$\omega(t) = |\{f > t\}|, \quad t \geq 0.$$ 

Assume that

$$\omega(t) \leq \frac{2}{1 + t^2}, \quad t \in [0, \infty).$$

For which values of $p > 0$ is $\int_E f^p$ finite?

(b) Find a set $E$ and a function $f : E \to \mathbb{R}$ showing the sharpness of your range of $p$ in part (a): that is, $\int_E f^p$ finite is precisely for the range of $p$ you found in part (a).

8. Assume that $g : [a, b] \to [c, d]$ and $f : [c, d] \to \mathbb{R}$ are each absolutely continuous. Prove that if $g$ is monotone increasing on $[a, b]$, then $f \circ g$ is absolutely continuous on $[a, b]$. 

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