Chapter 2

System of Linear Equations: Geometry

Where are we?

In Chapter 1 we learned to solve any system of linear equations in any number of variables. The answer is row reduction, which gives an algebraic solution. In Chapter 2 we put some geometry behind the algebra. It is the geometry that gives us intuition and deeper meaning. There are three main points:

Sec 2.3: $Ax = b$ is consistent $\Leftrightarrow$ b is in the span of the columns of A.

Sec 2.4: The solutions to $Ax = b$ are parallel to the solutions to $Ax = 0$.

Sec 2.9: The dim's of $\{b : Ax = b \text{ is consistent}\}$ and $\{\text{solutions to } Ax = b\}$ add up to the number of columns of A.

Section 2.1

Vectors

Outline

- Think of points in $\mathbb{R}^n$ as vectors.
- Learn how to add vectors and multiply them by a scalar
- Understand the geometry of adding vectors and multiplying them by a scalar
- Understand linear combinations algebraically and geometrically

Vectors

A vector is a matrix with one row or one column. We can think of a vector with n rows as:

- a point in $\mathbb{R}^n$
- an arrow in $\mathbb{R}^n$

To go from an arrow to a point in $\mathbb{R}^n$, we subtract the tip of the arrow from the starting point. Note that there are many arrows representing the same vector.

Adding vectors / parallelogram rule ▸ Demo

Scaling vectors ▸ Demo

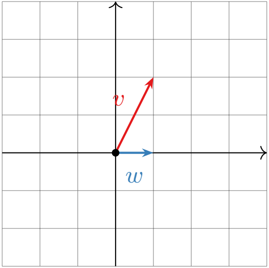
A scalar is just a real number. We use this term to indicate that we are scaling a vector by this number.

Linear Combinations

A linear combination of the vectors $v_1, \ldots, v_k$ is any vector

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

where $c_1, \ldots, c_k$ are real numbers.



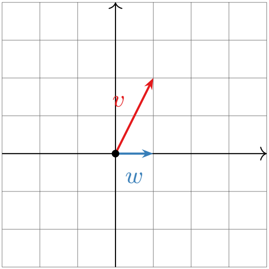
Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

Poll

Is there a vector in $\mathbb{R}^2$ that is not a linear combination of v and w?

- yes
- no



Linear Combinations

What are some linear combinations of $(1, 1)$?

What are some linear combinations of $(1, 1)$ and $(2, 2)$?

What are some linear combinations of $(0, 0)$?

Span

Essential vocabulary word!

$$
\begin{aligned}
\text{Span}\{v_1, v_2, \ldots, v_k\} &= \{x_1v_1 + x_2v_2 + \cdots + x_kv_k \mid x_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)} \\
&= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\
&= \text{plane through the origin and } v_1, v_2, \ldots, v_k.
\end{aligned}
$$

What are the possibilities for the span of two vectors in $\mathbb{R}^2$?

Demo

Span

Essential vocabulary word!

$$\begin{aligned}\text{Span}\{v_1, v_2, \ldots, v_k\} &= \{x_1 v_1 + x_2 v_2 + \cdots + x_k v_k \mid x_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

Demo

Conclusion: Spans are planes (of some dimension) through the origin, and the dimension of the plane is at most the number of vectors you started with and is at most the dimension of the space they're in.

Summary of Section 2.1

- A vector is a point/arrow in $\mathbb{R}^n$
- We can add/scale vectors algebraically & geometrically (parallelogram rule)
- A linear combination of vectors $v_1, \ldots, v_k$ is a vector

$$c_1 v_1 + \cdots + c_k v_k$$

  where $c_1, \ldots, c_k$ are real numbers.

Typical exam questions

True/False: For any collection of vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$, the zero vector in $\mathbb{R}^n$ is a linear combination of $v_1, \ldots, v_k$.

True/False: The vector $(1, 1)$ can be written as a linear combination of $(2, 2)$ and $(-2, -2)$ in infinitely many ways.

Describe geometrically the set of linear combinations of the vectors $(1, 0, 0)$ and $(1, 2, 3)$.

Suppose that v is a vector in $\mathbb{R}^n$, and consider the set of all linear combinations of v. What geometric shape is this?

True/False: It is possible for the span of 3 vectors in $\mathbb{R}^3$ to be a line.

True/False: the plane $z = 1$ in $\mathbb{R}^3$ is a span.