Section 3.2

One-to-one and onto transformations
Section 3.2 Outline

- Learn the definitions of one-to-one and onto functions
- Determine if a given matrix transformation is one-to-one and/or onto
One-to-one and onto in calculus

What do one-to-one and onto mean for a function $f : \mathbb{R} \to \mathbb{R}$?
One-to-one

A matrix transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is one-to-one if each \( b \) in \( \mathbb{R}^m \) is the output for at most one \( v \) in \( \mathbb{R}^n \).

In other words: different inputs have different outputs.

Do not confuse this with the definition of a function, which says that for each input \( x \) in \( \mathbb{R}^n \) there is at most one output \( b \) in \( \mathbb{R}^m \).
One-to-one

\( T : \mathbb{R}^n \to \mathbb{R}^m \) is one-to-one if each \( b \) in \( \mathbb{R}^m \) is the output for at most one \( v \) in \( \mathbb{R}^n \).

**Theorem.** Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:

- \( T \) is one-to-one
- the columns of \( A \) are linearly independent
- \( Ax = 0 \) has only the trivial solution
- \( A \) has a pivot in each column
- the range of \( T \) has dimension \( n \)

What can we say about the relative sizes of \( m \) and \( n \) if \( T \) is one-to-one?

Draw a picture of the range of a one-to-one matrix transformation \( \mathbb{R} \to \mathbb{R}^3 \).
A matrix transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **onto** if the range of $T$ equals the codomain $\mathbb{R}^m$, that is, each $b$ in $\mathbb{R}^m$ is the output for at least one input $v$ in $\mathbb{R}^m$. 
Onto

\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^m \] is onto if the range of \( T \) equals the codomain \( \mathbb{R}^m \), that is, each \( b \) in \( \mathbb{R}^m \) is the output for at least one input \( v \) in \( \mathbb{R}^n \).

**Theorem.** Suppose \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:

- \( T \) is onto
- the columns of \( A \) span \( \mathbb{R}^m \)
- \( A \) has a pivot in each row
- \( Ax = b \) is consistent for all \( b \) in \( \mathbb{R}^m \)
- the range of \( T \) has dimension \( m \)

What can we say about the relative sizes of \( m \) and \( n \) if \( T \) is onto?

Give an example of an onto matrix transformation \( \mathbb{R}^3 \rightarrow \mathbb{R} \).
One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

\[
\begin{pmatrix}
1 & 0 & 7 \\
0 & 1 & 2 \\
0 & 0 & 9
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & 1 & 1
\end{pmatrix}
\]
One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\text{ reflection}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\text{ projection}
\]

\[
\begin{pmatrix}
3 & 0 \\
0 & 3
\end{pmatrix}
\text{ scaling}
\]

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\text{ shear}
\]

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\text{ rotation}
\]
Which are one to one / onto?

Poll

Which give one to one-to-one / onto matrix transformations?

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 2 \\
-2 & 2 & -4
\end{pmatrix}
\]
Consider the robot arm example from the book.

There is a natural function \( f \) here (not a matrix transformation). The input is a set of three angles and the co-domain is \( \mathbb{R}^2 \). Is this function one-to-one? Onto?
The geometry

Say that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

The geometry of one-to-one:

The range has dimension $n$ (and the null space is a point).

The geometry of onto:

The range has dimension $m$, so it is all of $\mathbb{R}^m$ (and the null space has dimension $n - m$).
Summary of Section 3.2

- \( T : \mathbb{R}^n \to \mathbb{R}^m \) is one-to-one if each \( b \) in \( \mathbb{R}^m \) is the output for at most one \( v \) in \( \mathbb{R}^n \).

- **Theorem.** Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:
  - \( T \) is one-to-one
  - the columns of \( A \) are linearly independent
  - \( Ax = 0 \) has only the trivial solution
  - \( A \) has a pivot in each column
  - the range has dimension \( n \)

- \( T : \mathbb{R}^n \to \mathbb{R}^m \) is onto if the range of \( T \) equals the codomain \( \mathbb{R}^m \), that is, each \( b \) in \( \mathbb{R}^m \) is the output for at least one input \( v \) in \( \mathbb{R}^n \).

- **Theorem.** Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:
  - \( T \) is onto
  - the columns of \( A \) span \( \mathbb{R}^m \)
  - \( A \) has a pivot in each row
  - \( Ax = b \) is consistent for all \( b \) in \( \mathbb{R}^m \).
  - the range of \( T \) has dimension \( m \).
Typical exam questions

- True/False. It is possible for the matrix transformation for a $5 \times 6$ matrix to be both one-to-one and onto.
- True/False. The matrix transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by projection to the $yz$-plane is onto.
- True/False. The matrix transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by rotation by $\pi$ is onto.
- Is there an onto matrix transformation $\mathbb{R}^2 \rightarrow \mathbb{R}^3$? If so, write one down, if not explain why not.
- Is there an one-to-one matrix transformation $\mathbb{R}^2 \rightarrow \mathbb{R}^3$? If so, write one down, if not explain why not.