Section 3.3
Linear Transformations
Section 3.3 Outline

• Understand the definition of a linear transformation
• Linear transformations are the same as matrix transformations
• Find the matrix for a linear transformation
Linear transformations

A function \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear transformation if

- \( T(u + v) = T(u) + T(v) \) for all \( u, v \) in \( \mathbb{R}^n \).
- \( T(cv) = cT(v) \) for all \( v \) in \( \mathbb{R}^n \) and \( c \) in \( \mathbb{R} \).

First examples: matrix transformations.
Linear transformations

A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if

- $T(u + v) = T(u) + T(v)$ for all $u, v$ in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all $v$ in $\mathbb{R}^n$ and $c$ in $\mathbb{R}$.

Notice that $T(0) = 0$. Why?

We have the standard basis vectors for $\mathbb{R}^n$:

$$
e_1 = (1, 0, 0, \ldots, 0)$$
$$
e_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$

If we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$. Why?

In engineering, this is called the principle of superposition.
Which are linear transformations?
And why?

\[ T \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + y \\ y \\ x - y \end{array} \right) \]

\[ T \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + y + 1 \\ y \\ x - y \end{array} \right) \]

\[ T \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} xy \\ y \\ x - y \end{array} \right) \]

A function \( \mathbb{R}^n \to \mathbb{R}^m \) is linear exactly when the coordinates are linear (linear combinations of the variables, no constant terms).
Linear transformations

Which properties of a linear transformation fail for this function $T : \mathbb{R}^2 \to \mathbb{R}^2$?

$$T \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x \\ |y| \end{array} \right)$$
Linear transformations are matrix transformations

**Theorem.** Every linear transformation is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is an $m \times n$ matrix $A$ so that

$$T(v) = Av$$

for all $v$ in $\mathbb{R}^n$.

The matrix for a linear transformation is called the **standard matrix**.
Linear transformations are matrix transformations

**Theorem.** Every linear transformation is a matrix transformation.

Given a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ the standard matrix is:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{pmatrix}$$

Why? Notice that $Ae_i = T(e_i)$ for all $i$. Then it follows from linearity that $T(v) = Av$ for all $v$. 
The identity linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$T(v) = v$$

What is the standard matrix?

This standard matrix is called $I_n$ or $I$. 
Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function given by:

$$T \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x + y \\ y \\ x - y \end{pmatrix}$$

What is the standard matrix for $T$?
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the $x$-direction and 3 in the $y$-direction, and then reflects over the line $y = x$. 
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the $y$-axis and then rotates counterclockwise by $\pi/2$. 
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane.
Discussion Question

Find a matrix that does this.
A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
\begin{itemize}
    \item $T(u + v) = T(u) + T(v)$ for all $u, v$ in $\mathbb{R}^n$.
    \item $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and $c$ in $\mathbb{R}$.
\end{itemize}

**Theorem.** Every linear transformation is a matrix transformation (and vice versa).

The standard matrix for a linear transformation has its $i$th column equal to $T(e_i)$.
Typical Exam Questions Section 3.3

- Is the function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$ a linear transformation?
- Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and that

$$
T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}
$$

What is $T \begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

- Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates about the $z$-axis by $\pi$ and then scales by 2.
- Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the function given by:

$$
T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}
$$

Is this a linear transformation? If so, what is the standard matrix for $T$?
- Is the identity transformation one-to-one?