Section 3.4
Matrix Multiplication
Section 3.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things
Function composition

Remember from calculus that if \( f \) and \( g \) are functions then the composition \( f \circ g \) is a new function defined as follows:

\[
f \circ g(x) = f(g(x))
\]

In words: first apply \( g \), then \( f \).

Example: \( f(x) = x^2 \) and \( g(x) = x + 1 \).

Note that \( f \circ g \) is usually different from \( g \circ f \).
Composition of linear transformations

We can do the same thing with linear transformations $T : \mathbb{R}^p \to \mathbb{R}^m$ and $U : \mathbb{R}^n \to \mathbb{R}^p$ and make the composition $T \circ U$.

Notice that both have an $p$. Why?

What are the domain and codomain for $T \circ U$?

Natural question: What is the matrix for $T \circ U$? We’ll see!

Associative property: $(S \circ T) \circ U = S \circ (T \circ U)$

Why?
Composition of linear transformations

Example. \( T = \) projection to \( y \)-axis and \( U = \) reflection about \( y = x \) in \( \mathbb{R}^2 \)

What is the standard matrix for \( T \circ U \)?

What about \( U \circ T \)?
Matrix Multiplication
And now for something completely different (not really!)

Suppose $A$ is an $m \times n$ matrix. We write $a_{ij}$ or $A_{ij}$ for the $ij$th entry.

If $A$ is $m \times n$ and $B$ is $n \times p$, then $AB$ is $m \times p$ and

$$(AB)_{ij} = r_i \cdot b_j$$

where $r_i$ is the $i$th row of $A$, and $b_j$ is the $j$th column of $B$.

Or: the $j$th column of $AB$ is $A$ times the $j$th column of $B$.

Multiply these matrices (both ways):

$$
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\begin{pmatrix}
0 & -2 \\
1 & -1 \\
2 & 0
\end{pmatrix}
$$
Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do $U$ then do $T$

**Fact.** Suppose that $A$ and $B$ are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is $AB$.

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case $Bv$ is the $i$th column of $B$. So the left-hand side is $A$ times the $i$th column of $B$. The right-hand side is the $i$th column of $AB$ which we already said was $A$ times the $i$th column of $B$. It works!
Matrix Multiplication and Linear Transformations

**Fact.** Suppose that $A$ and $B$ are the standard matrices for the linear transformations $T : \mathbb{R}^p \to \mathbb{R}^m$ and $U : \mathbb{R}^n \to \mathbb{R}^p$. The standard matrix for $T \circ U$ is $AB$.

**Example.** $T =$ projection to $y$-axis and $U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane.
Discussion Question

Are there nonzero matrices $A$ and $B$ with $AB = 0$?

1. Yes
2. No
Properties of Matrix Multiplication

- \( A(BC) = (AB)C \)
- \( A(B + C) = AB + AC \)
- \( (B + C)A = BA + CA \)
- \( r(AB) = (rA)B = A(rB) \)
- \( (AB)^T = B^T A^T \)
- \( I_mA = A = AI_n \), where \( I_k \) is the \( k \times k \) identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- \( AB \) is not always equal to \( BA \)
- \( AB = AC \) does not mean that \( B = C \)
- \( AB = 0 \) does not mean that \( A \) or \( B \) is 0
More rabbits

Recall that the following matrix describes the change in our rabbit population from this year to the next:

\[
\begin{pmatrix}
0 & 6 & 8 \\
1/2 & 0 & 0 \\
0 & 1/2 & 0
\end{pmatrix}
\]

What matrix should we use if we want to describe the change in the rabbit population from this year to two years from now? Or 10 years from now?
Fun with matrix multiplication

Play the Buzz game!

http://textbooks.math.gatech.edu/ila/demos/transform_game.html

In the rotation game, you need to find a composition of shears that gives a rotation!
Summary of Section 3.4

- Composition: \((T \circ U)(v) = T(U(v))\) (do \(U\) then \(T\))
- Matrix multiplication: \((AB)_{ij} = r_i \cdot b_j\)
- Matrix multiplication: the \(i\)th column of \(AB\) is \(A(b_i)\)
- Suppose that \(A\) and \(B\) are the standard matrices for the linear transformations \(T : \mathbb{R}^n \rightarrow \mathbb{R}^m\) and \(U : \mathbb{R}^p \rightarrow \mathbb{R}^n\). The standard matrix for \(T \circ U\) is \(AB\).
- Warning!
  - \(AB\) is not always equal to \(BA\)
  - \(AB = AC\) does not mean that \(B = C\)
  - \(AB = 0\) does not mean that \(A\) or \(B\) is 0
Typical Exam Questions 3.4

- True/False. If $A$ is a $3 \times 4$ matrix and $B$ is a $4 \times 3$ matrix, then it makes sense to multiply $A$ and $B$ in both orders.
- True/False. If it makes sense to multiply a matrix $A$ by itself, then $A$ must be a square matrix.
- True/False. If $A$ is a non-zero square matrix, then $A^2$ is a non-zero square matrix.
- True/False. If $A = -I_n$ and $B$ is an $n \times n$ matrix, then $AB = BA$.
- Find the standard matrices for the projections to the $xy$-plane and the $yz$-plane in $\mathbb{R}^3$. Find the matrices for the linear transformations obtained by doing these two linear operations in the two different orders. Are the answers the same?
- Find the standard matrix $A$ for projection to the $xy$-plane in $\mathbb{R}^3$. What is $A^2$?
- Find the standard matrix $A$ for reflection in the $xy$-plane in $\mathbb{R}^3$. Is there a matrix $B$ so that $AB = I_3$?