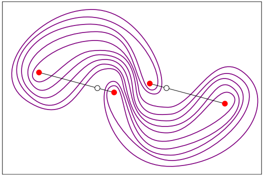
Taffy pullers

How efficient is this taffy puller?



If you run the taffy puller, the taffy starts to look like the shape on the right. Every rotation of the machine changes the number of strands of taffy by a matrix:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

The largest eigenvalue λ of this matrix describes the efficiency of the taffy puller. With every rotation, the number of strands multiplies by λ.

Section 5.5

Complex Eigenvalues

Outline of Section 5.5

- Rotation matrices have no eigenvectors
- Crash course in complex numbers
- Finding complex eigenvectors and eigenvalues
- Complex eigenvalues correspond to rotations + dilations

▸ Demo ▸ Demo

A matrix without an eigenvector

Recall that rotation matrices like

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

have no eigenvectors. Why?

Imaginary numbers

Problem. When solving polynomial equations, we often run up against the issue that we can't take the square root of a negative number:

$$x^2 + 1 = 0$$

Solution. Take square roots of negative numbers:

$$x = \pm\sqrt{-1}$$

We usually write $\sqrt{-1}$ as i (for "imaginary"), so $x = \pm i$.

Now try solving these:

$$x^2 + 3 = 0$$

$$x^2 - x + 1 = 0$$

Complex numbers

We can add/multiply (and divide!) complex numbers:

$(2 - 3i) + (-1 + i) =$

$(2 - 3i)(-1 + i) =$

Complex numbers

The complex numbers are the numbers

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can conjugate complex numbers: $\overline{a + bi} = a - bi$

Complex numbers and polynomials

Fundamental theorem of algebra. Every polynomial of degree n has exactly n complex roots (counted with multiplicity).

Fact. If z is a root of a real polynomial then $\overline{z}$ is also a root.

So what are the possibilities for degree 2, 3 polynomials?

What does this have to do with eigenvalues of matrices?

Complex eigenvalues

Say A is a square matrix with real entries.

We can now find complex eigenvectors and eigenvalues.

Fact. If λ is an eigenvalue of A with eigenvector v then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{v}$.

Why?

Trace and determinant

Now that we have complex eigenvalues, we have the following fact.

Fact. The sum of the eigenvalues of A (with multiplicity) is the trace of A and the product of the eigenvalues of A (with multiplicity) is the determinant.

Indeed, by the fundamental theorem of algebra, the characteristic polynomial factors as:

$$(x_1 - \lambda)(x_2 - \lambda) \cdots (x_n - \lambda).$$

From this we see that the product of the eigenvalues $x_1 x_2 \cdots x_n$ is the constant term, which we said was the determinant, and the sum $x_1 + x_2 + \cdots + x_n$ is $(-1)^{n-1}$ times the λ^{n-1} term, which we said was the trace.

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Three shortcuts for complex eigenvectors

Suppose we have a 2×2 matrix with complex eigenvalue λ.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

(3) Also, we get the other eigenvalue/eigenvector pair for free: conjugation.

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

Summary of Section 5.5

- Complex numbers allow us to solve all polynomials completely, and find n eigenvalues for an $n \times n$ matrix, counting multiplicity
- If λ is an eigenvalue with eigenvector v then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$

Typical Exam Questions 5.5

- True/False. If v is an eigenvector for A with complex entries then $i \cdot v$ is also an eigenvector for A.
- True/False. If $(i, 1)$ is an eigenvector for A then $(i, -1)$ is also an eigenvector for A.
- If A is a 4×4 matrix with real entries, what are the possibilities for the number of non-real eigenvalues of A?
- Find the eigenvalues and eigenvectors for the following matrices.

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$