

Section 5.6

Stochastic Matrices (and Google!)

Outline of Section 5.6

- Stochastic matrices and applications
- The steady state of a stochastic matrix
- Important web pages

Stochastic matrices

A stochastic matrix is a non-negative square matrix where all of the columns add up to 1.

Examples:

$$\begin{pmatrix} 1/4 & 3/5 \\ 3/4 & 2/5 \end{pmatrix} \quad \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix} \quad \begin{pmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 0 & 1/4 \\ 0 & 0 & 1/4 \end{pmatrix}$$

Application: Rental Cars (or Redbox...)

Say your car rental company has 3 locations. Make a matrix whose ij entry is the fraction of cars at location j that end up at location i . For example,

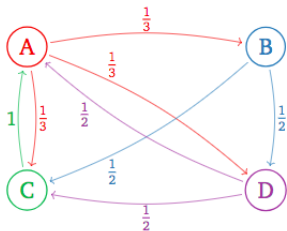
$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

Note the columns sum to 1. Why?

If there are 100 cars at each location on the first day, and every car gets rented, how many cars are at each location on the second day? third day? n th day?

Application: Web pages

Make a matrix whose ij entry is the fraction of (randomly surfing) web surfers at page j that end up at page i . If page i has N links then the ij -entry is either 0 or $1/N$.



$$\begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Which web page seems most important?

Properties of stochastic matrices

Let A be a stochastic matrix.

Fact 1. One of the eigenvalues of A is 1 and all other eigenvalues have absolute value at most 1.

Why?

Positive stochastic matrices

Let A be a **positive** stochastic matrix, meaning all entries are positive.

Fact 2. The 1-eigenspace of A is 1-dimensional; it has a positive eigenvector.

The unique such eigenvector with entries adding to 1 is called the **steady state vector**.

Example. If $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a 1-eigenvector, what's the steady state vector?

Example

Find the steady state vector.

$$A = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix}$$

More about positive stochastic matrices

Let A be a **positive** stochastic matrix, meaning all entries are positive.

Fact 3. Under iteration, all nonzero vectors approach a multiple of the steady state vector. The multiple is the sum of the entries of the original vector.

▶ Demo

The last fact tells us how to distribute rental cars, and also tells us the importance of web pages!

Example

To what vector does $A^n \begin{pmatrix} 1 \\ 9 \end{pmatrix}$ approach as $n \rightarrow \infty$

$$A = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix}$$

Application: Rental Cars

The rental car matrix is:

$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

Its steady state vector is:

$$\begin{pmatrix} 7/18 \\ 6/18 \\ 5/18 \end{pmatrix} \approx \begin{pmatrix} .39 \\ .33 \\ .28 \end{pmatrix}$$

Application: Web pages

The web page matrix is:

$$\begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Its steady state vector is approximately

$$\begin{pmatrix} .39 \\ .13 \\ .29 \\ .19 \end{pmatrix}$$

and so the first web page is the most important.

Fine print

There are a couple of problems with the web page matrix as given:

- What happens if there is a web page with no links?
- What if the internet graph is not connected?
- How do you find eigenvectors for a huge matrix?

Here are the solutions:

- Make a column with $1/n$ in each entry (the surfer goes to a new page randomly).
- Let B be the matrix with all entries equal to 1, replace A with

$$.85 * A + .15 * B$$

- Approximate via iteration!

Summary of Section 5.6

- A stochastic matrix is a non-negative square matrix where all of the columns add up to 1.
- Every stochastic matrix has 1 as an eigenvalue, and all other eigenvalues have absolute value at most 1.
- A positive stochastic matrix has 1-dimensional eigenspace and has a positive eigenvector. A positive 1-eigenvector with entries adding to 1 is called a steady state vector.
- For a positive stochastic matrix, all nonzero vectors approach the steady state vector under iteration.
- Steady state vectors tell us the importance of web pages (for example).

Typical Exam Questions 5.6

- Is there a stochastic matrix where the 1-eigenspace has dimension greater than 1?
- Find the steady state vector for this matrix:

$$A = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 1/3 \end{pmatrix}$$

To what vector does $A^n \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ approach as $n \rightarrow \infty$?

- Find the steady state vector for this matrix:

$$A = \begin{pmatrix} 1/3 & 1/5 & 1/4 \\ 1/3 & 2/5 & 1/2 \\ 1/3 & 2/5 & 1/4 \end{pmatrix}$$

- Make your own internet and see if you can guess which web page is the most important. Check your answer using the method described in this section.