Section 6.3
Orthogonal projection
Outline of Section 6.3

- Orthogonal projections and distance
- A formula for projecting onto any subspace
- A special formula for projecting onto a line
- Matrices for projections
- Properties of projections
Orthogonal Projections

Let $b$ be a vector in $\mathbb{R}^n$ and $W$ a subspace of $\mathbb{R}^n$.

The **orthogonal projection** of $b$ onto $W$ is the vector obtained by drawing a line segment from $b$ to $W$ that is perpendicular to $W$.

**Fact.** The following three things are all the same:

- The orthogonal projection of $b$ onto $W$
- The vector $b_W$ (the $W$-part of $b$)
- The closest vector in $W$ to $b$

**algebra!**

**geometry!**
Orthogonal Projections

**Theorem.** Let $W = \text{Col}(A)$. For any vector $b$ in $\mathbb{R}^n$, the equation

$$A^T Ax = A^T b$$

is consistent and the orthogonal projection $b_W$ is equal to $Ax$ where $x$ is any solution.
Orthogonal Projections

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Why? Choose $\hat{x}$ so that $A\hat{x} = b_W$. We know $b - b_W = b - A\hat{x}$ is in $W^\perp = \text{Nul}(A^T)$ and so

$$0 = A^T(b - A\hat{x}) = A^T b - A^T A\hat{x}$$

$$\Rightarrow A^T A\hat{x} = A^T b$$
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What does the theorem give when $W = \text{Span}\{u\}$ is a line?
Orthogonal Projection onto a line

Special case. Let $L = \text{Span}\{u\}$. For any vector $b$ in $\mathbb{R}^n$ we have:

$$b_L = \frac{u \cdot b}{u \cdot u}$$

Find $b_L$ and $b_{L\perp}$ if $b = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ and $u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. 
Orthogonal Projections

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is consistent and the orthogonal projection $b_W$ is equal to $Ax$ where $x$ is any solution.

Example. Find $b_W$ if $b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T b$, then solve for $x$, then compute $Ax$.

Question. How far is $b$ from $W$?
Orthogonal Projections

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is consistent and the orthogonal projection $b_W$ is equal to $Ax$ where $x$ is any solution.

**Special case.** If the columns of $A$ are independent then $A^T A$ is invertible, and so

$$b_W = A(A^T A)^{-1} A^T b.$$ 

Why? The $x$ we find tells us which linear combination of the columns of $A$ gives us $b_W$. If the columns of $A$ are independent, there's only one linear combination.
Matrices for projections

**Fact.** If the columns of $A$ are independent and $W = \text{Col}(A)$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal projection onto $W$ then the standard matrix for $T$ is:

$$A(A^T A)^{-1} A^T.$$ 

Why?

**Example.** Find the standard matrix for orthogonal projection of $\mathbb{R}^3$ onto $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

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Projections as linear transformations

Let $W$ be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Then

- $T$ is a linear transformation
- $T(b) = b$ if and only if $b$ is in $W$
- $T(b) = 0$ if and only if $b$ is in $W^\perp$
- $T \circ T = T$
- The range of $T$ is $W$
Properties of projection matrices

Let $W$ be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Let $A$ be the standard matrix for $T$. Then

- The 1–eigenspace of $A$ is $W$ (unless $W = 0$)
- The 0–eigenspace of $A$ is $W^\perp$ (unless $W = \mathbb{R}^n$)
- $A^2 = A$
- $\text{Col}(A) = W$
- $\text{Nul}(A) = W^\perp$
- $A$ is diagonalizable; its diagonal matrix has $m$ 1’s & $n - m$ 0’s where $m = \dim W$ (this gives another way to find $A$)

You can check these properties for the matrix in the last example. It would be very hard to prove these facts without any theory. But they are all easy once you know about linear transformations!
Summary of Section 6.3

- The **orthogonal projection** of \( b \) onto \( W \) is \( b_W \)
- \( b_W \) is the closest point in \( W \) to \( b \).
- The distance from \( b \) to \( W \) is \( \|b_W \perp\| \).
- **Theorem.** Let \( W = \text{Col}(A) \). For any \( b \), the equation \( A^T Ax = A^T b \) is consistent and \( b_W \) is equal to \( Ax \) where \( x \) is any solution.
- **Special case.** If \( L = \text{Span}\{u\} \) then \( b_L = \frac{u \cdot b}{u \cdot u} u \)
- **Special case.** If the columns of \( A \) are independent then \( A^T A \) is invertible, and so \( b_W = A(A^T A)^{-1} A^T b \)
- When the columns of \( A \) are independent, the standard matrix for orthogonal projection to \( \text{Col}(A) \) is \( A(A^T A)^{-1} A^T \)
- Let \( W \) be a subspace of \( \mathbb{R}^n \) and let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be the function given by \( T(b) = b_W \). Then
  - \( T \) is a linear transformation
  - etc.
- If \( A \) is the standard matrix then
  - The 1–eigenspace of \( A \) is \( W \) (unless \( W = 0 \))
  - etc.
Typical Exam Questions 6.3

• True/false. The solution to $A^T A x = A^T b$ is the point in $\text{Col}(A)$ that is closest to $b$.

• True/false. If $v$ and $w$ are both solutions to $A^T A x = A^T b$ then $v - w$ is in the null space of $A$.

• True/false. If $A$ has two equal columns then $A^T A x = A^T b$ has infinitely many solutions for every $b$.

• Find $b_L$ and $b_{L\perp}$ if $b = (1, 2, 3)$ and $L$ is the span of $(1, 2, 1)$.

• Find $b_W$ if $b = (1, 2, 3)$ and $W$ is the span of $(1, 2, 1)$ and $(1, 0, 1)$. Find the distance from $b$ to $W$.

• Find the matrix $A$ for orthogonal projection to the span of $(1, 2, 1)$ and $(1, 0, 1)$. What are the eigenvalues of $A$? What is $A^{100}$?