MATH 4803 - MAR
Intro. Geometric Group Theory
Spring 2021 GATECH
Basic Info

Canvas
- Piazza
- Gradescope
- Web page

1st HW

Assessments
- 10% Participation (Piazza)
- 30% HW
- 30% Midterm
- 30% Final project

Calendar

HW

Notes

Resources/Refs

Syllabus, Final Project
What is GGT?

Groups are collections of symmetries of geometric objects.

Use the geometry to learn about the algebraic properties of the group.
A group:

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It is $D_4$
A group:

\[ SL_2(\mathbb{Z}) = \{ 2 \times 2 \text{ integer matrices of } \}
\]
\[ \text{det} = 1 \]

Examples:

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

Lin alg: eigenvals, eigenvectors, ...
   (in transf. of \( \mathbb{R}^2 \))

Group theory:

generators?  
relations?  
torsion?  
subgroups?  
quotients?
A geometric object: Farey graph

Not obvious but true:
• all primitive integer vectors are vertices.
• each $A \in SL_2(\mathbb{Z})$ gives a symmetry of the graph.

Check: if $v, w$ connected by edge then $Av, Aw$ connected by edge $(0, -1)$ takes $(1)$ to $(0, -1)(1) = (-1)$

Rotation by $\pi$. 
To do the check, prove:

\((p, q) \& (r, s)\) are connected by an edge

\(\iff \det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1.\)

Check: if \(v, w\) connected by edge

then \(A(v), A(w)\) connected by edge

\[ A\left( \begin{pmatrix} p & r \\ q & s \end{pmatrix} \right) = \left( A\left( \begin{pmatrix} p \\ q \end{pmatrix} \right), A\left( \begin{pmatrix} r \\ s \end{pmatrix} \right) \right) \]

If \((p, q) \rightarrow (r, s)\) then \(\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1\)

then \(\det A\left( \begin{pmatrix} p \\ q \end{pmatrix} \right) = \pm 1\) then \(A\left( \begin{pmatrix} p \\ q \end{pmatrix} \right) \rightarrow A\left( \begin{pmatrix} r \\ s \end{pmatrix} \right)\)
Overview of Course

Chap 1. Cayley graph

$G \leftrightarrow$ graph

$\mathbb{Z}$

$\mathbb{Z}^2$

$\mathbb{Z}$ with gen set $\{3, 2\}$

7/15/71

0 ends

2 ends

1 end

00 ends

-3 -2 -1 0 1 2 3 4
Chap 2 Coxeter gps
= groups gen by reflections

e.g., $\mathbb{R}^2$

example

Chap 3 Groups acting on trees

Free gps

$F_n =$ gp with $n$ gens & no relations

- Groups acting (freely) on trees $\leftrightarrow$ free gps

- $F_3 \leq F_2$
- $F_\infty \leq F_2$
Chap 4  Baumslag-Solitar gps
see pic on web.

Chap 4  Baumslag-Solitar gps
see pic on web.

hyperbolic plane. "treelike"

Chap 5  Word problem
Given a product of gems, is it id in gp.

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Given a product of gems, is it id in gp.

Chap 8  Lamplighter gp

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Chap 10  Thompson's gp.

Chap 10  Thompson's gp.

Chap 11  Large scale
properties

Chap 11  Large scale
properties

Thm. Every gp has
0, 1, 2, or 00 many
ends.
Announcements Jan 19

- Cameras on in class
- 1st HW assigned Thu, due Tue 3:30.
- Lecture notes/HW posted on web site.
- Groups/topics due Feb 5
- Office hours Tue 11-12, Fri 2-3, appt

Gradescope

Q. How many symmetries does a cube have? tetrahedron, icosahedron, ...?
Groups

$G$ set

$G \times G \rightarrow G$ mult.

$\exists \text{id}$

$\exists \text{inv.}$

$\exists$ assoc.

example: symmetries of... anything.

Examples of finite groups

1. Dihedral group $D_n$

   = set of symmetries of $n$-gon

   $s, t$ are generators.

   since $st$ is a rotation

   relations:

   $s^2 = t^2 = \text{id}$

   $(st)^n = \text{id}$

   This is a presentation for $D_n$:

   $\langle s, t \mid s^2 = t^2 = (st)^n = \text{id} \rangle$
2) Symmetric group.

$S_n =$ set of permutations of $\{1, \ldots, n\}$.

generators: $(i \ i+1) = \tau_i$

$\tau$ gives a presentation for $S_n$!

$\tau_1 \tau_2 \tau_1 \tau_2 \tau_1$ for $S_n$.

Generators: $\tau_1, \ldots, \tau_{n-1}$

Relations: $\tau_i^2 = 1$.

$\tau_i \tau_j = \tau_j \tau_i$ if $|i-j| > 1$

$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$

$i = 1, \ldots, n-2$
③ Finite cyclic gps

\[ \mathbb{Z}/n\mathbb{Z} \]

What is it the symmetries of?

n-gon.

Presentation?

\[ \langle a \mid a^n = \text{id} \rangle \]

④ Trivial group

\[ \langle 1 \rangle \text{ or } \langle a \mid a \rangle \]

Examples of Infinite groups

① \[ \mathbb{Z} \]

\[ \langle a \rangle \]

\[ a, a^2, a^3, a^{-3} \]

② \[ \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{(a, b) : a, b \in \mathbb{Z}\} \]

\[ \langle a, b \mid ab = ba \rangle \]

\[ a = (1,0), \quad b = (0,1) \]

\[ aba = a^2 b = (2,1) \]
$SL_n \mathbb{Z} = \{ n \times n \text{ integer matrices with } \det = 1 \}$

What is this the symmetries of?

Presentation?

Harder!
Free groups

$F_2 = \{ \text{freely reduced words in } a, b \}$

word: sequence of $a, b, a^{-1}, b^{-1}$

freely reduced: no $aa^{-1}, a^{-1}a$

$bb^{-1}, b^{-1}b$

Finite

Multiplication: concatenate, then freely reduce

e.g. $aba^{-1} \cdot ab = abb$

Check this is a group.

$id = \text{empty word}$

inverse = reverse & invert letters

e.g. $(abab)^{-1} = b^{-1}a^{-1}b^{-1}$

assoc.

An issue: different reductions lead to same reduced word.

e.g. $aa^{-1}bb^{-1}$

or $b^{-1}aa^{-1}bb^{-1}$

Presentation: $\langle a, b | \rangle$
So...

\[ \mathbb{Z} \cong F_1 \]

& \( F_0 \) = trivial group.

Later in the class:

In \( \text{SL}_2(\mathbb{Z}) \):

\[ a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \]

generate a free group.

so: \( a^5b^7a^{-1}b^{-4}a \neq \text{id} \).

Next time: Free gps are important because every countable group is a quotient of a free group.
Announcements Jan 21

- Please turn cameras on
- HW1 due Tue 3:30 (I need to set up Gradescope)
- HW1 Lecture notes posted on web site.
- Groups/topics due Feb 5
- Office hours Fri 2-3, Tue 11-12, by appt.

Need to add a reading prompt
Examples of groups

\( D_n, S_n, \mathbb{Z}/n \)

\( \mathbb{Z}^n, \text{SL}_n\mathbb{Z}, F_n \)

What is \( F_n \) the symmetries of \( \mathbb{F}_2 \)?

Braid groups \( B_n \)

elements:

\[ a^2 \]

up to isotopy:

\[ \begin{align*}
\text{multipl: concatenation} \\
B_3 \quad a \cdot b = ab
\end{align*} \]
Internal presentations

\[ \langle S \mid R \rangle \text{ is an internal presentation of } G \]

if

1. \( S \) is a generating set for \( G \)
2. If two words in \( SUS^{-1} \) are equal in \( G \), they differ by a finite seq. of elements of \( R \cup \{ss^{-1} : s \in SUS^{-1}\} \)

(replacing one side of an equality with another)

Fact. Every group has one: \( S = G \)
\( R = \text{ every possible equality.} \)
Example, $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} : \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1 \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$

$\sigma_i \quad \sigma_{i+1} \quad \sigma_i$
Homomorphisms

\[ f: G \rightarrow H \]
\[ f(ab) = f(a)f(b) \]

Injective homomorphisms

“Putting one group into another as a subgp”

- \( \mathbb{Z}/n \rightarrow D_n \) rotations.
- \( \mathbb{Z}/2 \rightarrow D_n \) reflection.
- \( \mathbb{Z} \rightarrow F_2 \)
  \[ 1 \mapsto a \]

Which groups have inj homoms to \( B_n \)?

\[ \mathbb{Z} \quad 1 \mapsto \sigma_1 \quad \checkmark \]

\[ \mathbb{Z}^2 \quad (1,0) \mapsto \sigma_1 \quad \text{assuming} \quad n > 4 \]

\[ \mathbb{Z}/2 \quad \text{Does } B_n \text{ have an elt of order 2?} \quad \text{No.} \]

\[ F_2 \quad a \mapsto \sigma_1^2 \]
\[ b \mapsto \sigma_2^2 \]
Non-injective homs

"forgetting (wisely)"

\[ \mathbb{Z} \to \mathbb{Z}/2 \quad \text{even/odd} \]

\[ \mathbb{Z} \to \mathbb{Z}/10 \quad 1's \ digit \]

\[ D_n \to \mathbb{Z}/2 \quad \text{flip?} \]

\[ F_2 \to \mathbb{Z}^2 \quad \text{exponent} \]

\[ a \mapsto (1,0), \quad b \mapsto (0,1) \quad \text{sum on} \ a \ & b \]

\[ B_n \to S_n \]

---

Normal subgps \( N \leq G \)

\[ gNg^{-1} = N \quad \forall g \in G \]

\[ \{ \text{kernels} \} \leftrightarrow \{ \text{normal subps} \} \quad \text{of } G \]

First Isomorphism Thm

If \( f : G \to H \) surj homom. with kernel \( K \)

Then \( H \cong G/K \).
**First Isomorphism Thm**

If \( f: G \to H \) surj homom. with kernel \( K \), then \( H \cong G/K \).

**Surj. homoms**

- \( \mathbb{Z} \to \mathbb{Z}/2 \)
- \( \mathbb{Z} \to \mathbb{Z}/10 \)
- \( D_n \to \mathbb{Z}/2 \)
- \( F_2 \to \mathbb{Z}^2 \)
- \( a \to (1,0) \)
- \( b \to (0,1) \)
- \( B_n \to S_n \)

\( \mathbb{Z}/2 \mathbb{Z} \cong \mathbb{Z}/2 \)

**1st Isom. Thm**

- \( D_n/\text{rotations} \cong \mathbb{Z}/2 \)
- \( D_n/74n \cong \mathbb{Z}/2 \)

\( F_2/F_2' = \mathbb{Z}^2 \)

\( L \text{ commutator subgp.} \)

\( B_n/\text{PB}_n \cong S_n \)
External presentation

\[ \langle S \mid R \rangle \]

\( S = \) set of \( R \): equalities between words in \( S \) and \( S^{-1} \) and \( \text{id} \).

Not \( ab=ba \), but \( aba^{-1}b^{-1} = \text{id} \)

We obtain a group: \( F(S)/\langle \langle R \rangle \rangle \)

Free gp on \( S \)

HW. Internal & External presentations are equivalent.

Consequence. Every gp is a quotient of a free group.

\( H = \langle a, b \mid aba^{-1}b^{-1}=\text{id}\rangle = F^2/\langle \langle K \rangle \rangle \)
Announcements Jan 26

- Cameras on for class
- HW1 due Thu 3:30
- Groups/topics due Feb 5
- Office hours Fri 2-3, Tue 11-12, appt

How many symmetries does a cube have?

\[ \langle a, b, c, d \mid ab^{-1}c^{-1}d^{-1}\rangle \]
Symmetries

\[ X = \text{math object} \]
\[ \text{Sym}(X) = \{ \text{symmetries of } X \} \]

- group under composition.

**Examples**

\[ \begin{array}{ccc}
\text{regular n-gon} & \{1, \ldots, n\} & \text{n-dim vector space over } F \\
D_n & S_n & \text{GL}_n F
\end{array} \]

\[ \mathbb{R}^2 \text{ as a vector space} \]
\[ \text{Euclidean 2-space} \]
\[ \text{topological space} \]
\[ \text{group } G \]

\[ \text{GL}_2 \mathbb{R} \]
\[ \text{Aff}(\mathbb{R}^2) \]
\[ \text{line-preserving} \]
\[ \text{Homeo}(X) \]
\[ \text{Aut}(G) \]
**Actions**

An action of a group $G$ on a math object $X$ is a homomorphism

$$G \rightarrow \text{Sym}(X)$$

or a map

$$G \times X \rightarrow X$$

with

$$e \cdot x = x \quad \forall \ x \in X$$

$$g \cdot (h \cdot x) = (gh) \cdot x \quad \forall \ g, h \in G, \ x \in X$$

and the restriction

$$g \times X \rightarrow X$$

is in $\text{Sym}(X)$ $\forall \ g \in G$

**Examples**

- $D_n \subset \mathbb{C}$: $n$-gon (filled in or not)
- $\mathbb{R}^2$: $\{\text{vertices of } n\text{-gon}\}$
- $\mathbb{R}^2$: $\{\text{diagonals of } n\text{-gon}\}$
- $\text{SL}_2 \subset \mathbb{C}$: $\mathbb{R}^2$ as a vector space
- $\mathbb{C}$: $\{\text{vectors in } \mathbb{R}^2\}$
- $\mathbb{C}$: $\{\text{primitive vectors...}\}$
- $\mathbb{C}$: Farey graph
Two vocab words:
1. If have $G \leq X$, say $G$ is represented by symmetries of $X$.
2. An action is faithful if $G \to \text{Sym}(X)$ is injective.

Cayley's Thm

Every group can be represented as a group of permutations faithfully.

Rephrase: there is $G \to \text{Sym}(X)$ for $X$ a set.

If

Take $X = G$ as a set. $\to \text{Sym}(X)$ is a permutation group.

Given $g \in G$ need a permutation of $X = G$.

$G \times G \to G$

$(g, h) \mapsto gh$

Need to check: defn of action. $\checkmark$ faithful.
Examples

$G = \mathbb{Z}^2$

$X = \mathbb{Z}^2$

The action of $(2,1)$ on $\mathbb{Z}^2$
Graphs

A graph $\Gamma$ is a set $V(\Gamma)$, a set $E(\Gamma)$, and a function

$$\text{Ends} : E(\Gamma) \to \{ \{u, v\} : u, v \in V(\Gamma) \}$$

Examples

$K_n = \text{complete graph on } n \text{ vertices}$

$K_{m,n} = \text{complete bipartite graph ...}$

Tree = connected graph with no cycles.
Symmetries of graphs

A symmetry of a graph $\Gamma$ is a pair of bijections

$\alpha : V(\Gamma) \rightarrow V(\Gamma)$

$\beta : E(\Gamma) \rightarrow E(\Gamma)$

preserving Ends function:

$\text{Ends}(\beta(e)) = \alpha \cdot \text{Ends}(e)$

Examples

$\text{Sym}(K_n) \cong \text{Sym}_n \cong \text{Sym}(V(K_n))$

$\text{Sym}(K_{m,n}) \cong \text{Sym}_m \times \text{Sym}_n$

unless $m = n$.

How many symmetries? 12

$\text{deg} = 3 \quad \text{permute}$

$\text{deg} = 2$

$\text{Sym}_3 \times \mathbb{Z}/2$
Graphs can have "decorations":
- directed edges
- labeled edges

"directed graph"

\[ \text{Sym}^+(\Gamma) = \{ \text{symmetries of } \Gamma \text{ preserving decorations} \} \]

Cayley's better theorem: Every group is faithfully represented as symmetries of a graph. (next time)
Cayley graphs

\[ G = \text{group} \]
\[ S = \text{gen set} \]

\[ \Rightarrow \text{Cayley graph for } G \text{ with respect to } S \]

has vertices: \[ G \]
edges: \[ g \rightarrow gs \]
\[ g \in G \]
\[ s \in S \]

Examples

1. \( G = \mathbb{Z}/n \), \( S = \{1\} \)
2. \( G = \mathbb{Z}^2 \), \( S = \{(1,0), (0,1)\} \)
Announcements Jan 28

- Cameras on
- HW 2 due next week Thu 3:30
- Groups/topics due Feb 5
- Office hours Fri 2-3, Tue 11-12, appt
- Way-too-early course evals
Goals today:

- Given a graph, what are its symmetries?
- Given a group, what graph(s) does it act on?

Why? e.g. Thm. If $G$ acts freely on a tree then $G$ is free.
Last time: (1) Graphs

Exactly 2 symmetries.

Symmetries: permutation of $V(\Gamma)$, $E(\Gamma)$ respecting Ends.

(2) Actions

$G \rightarrow X$

means: $G \xrightarrow{} \text{Sym}(X)$

or $G \times X \rightarrow X$

$(g, x) \xrightarrow{} g \cdot x$
1.4 Orbits & stabilizers

Say $G \times X$

$\text{Stab}(x) = \{ g \in G : g \cdot x = x \}$

this is a subgroup

e.g. $D_n \times n$-gon.

$\text{Stab}(v) \cong \mathbb{Z}/2$

$\cdot D_n \times n$-gon

$\text{Stab}(c) = D_n$

Orbit of $x$:

$\text{Orb}(x) = \{ g \cdot x : g \in G \}$

eg. $D_n \times n$-gon.

$|\text{Orb}(x)| = 2n$

$|\text{Orb}(v)| = n$

$|\text{Orb}(c)| = 1$

The action of $G$ is _free_ if $\text{Stab}(x) = \{ e \}$
Theorem. There is a bijection:

\[ \text{Orb}(x) \leftrightarrow \text{left cosets of Stab}(x) \]

given by \( g \cdot x \leftrightarrow g \text{Stab}(x) \).

Proof. Subtlety: well-definedness.

But \( g \cdot x = h \cdot x \iff h^{-1}g \cdot x = x \)

\[ \iff h^{-1}g \in \text{Stab}(x) \iff g \text{Stab}(x) = h\text{Stab}(x) \square \]

Corollary (Orbit-Stab Thm)

If \( |G| < \infty \) & \( G \circ x \), then

\[ |G| = |\text{Stab}(x)| \cdot |\text{Orb}(x)| \]

If. Lagrange's thm

Example. How many symmetries does a cube have?

\[ |G| = 3 \cdot 8 = 24 \text{ rotations} \]

or \( 6 \cdot 8 = 48 \text{ all symmetries} \)

Cor² If \( \text{Stab}(x) = \{e\} \) then:

\[ G \leftrightarrow \text{Orb}(x) \]
1.5 Cayley graphs

\[ G \quad S = \text{gen set.} \]

\[ \Gamma_{G,s} \]

\[ V(\Gamma_{G,s}) = G \]

\[ E(\Gamma_{G,s}) : \]

\[ g \rightarrow gS \]

\[ \forall g \in G, \quad s \in S \]

**Fact.** \( \Gamma_{G,s} \) is connected. (as undirected graph)

**Why?** Enough to show all vertices are connected by a path to \( e \).

Given a vertex \( g \), write as a product \( g = s_1 s_2 \ldots s_n, \quad s_i \in S \).

**Fact.** \( G \rightarrow \Gamma_{G,s} \)

As a labeled directed graph

On vertices: \( g \cdot h = gh \)

This extends to edges:

\[ hs \rightarrow g \hspace{1cm} ghs \rightarrow s \]

Fact. The action is faithful.
Examples

1. $D_3 = \text{Sym}_3$
   $S = \{ r, s \}$

$D_3 = \text{Sym}_3$

$S = \{ s, t \}$

Exercise...
$\mathbb{Z}^2$

$S = \{a, b\}$

$(1,0) \quad (0,1)$

or

$\mathbb{Z}$

$S = \{1\}$

$\mathbb{Z}$

$S = \{2, 3\}$

$F_2 \quad S = \{a, b\}$

Cycles in $\Gamma_{G, S}$

$\leftrightarrow$ relations.
Announcements Feb 2

- Cameras on
- HW 2 due Thu 3:30
- Groups/topics due Feb 5
- OH Fri 2-3, appt
- Way-too-early course feedback Canvas → Quizzes
Cayley graphs

$G = \text{group}$

$S = \text{genset}$

$\Gamma_{G, S} = \text{graph}$

Vertices: $G$

Edges: $\forall g \in G$

$\forall s \in S.$

Last time: $G \subset \Gamma_{G, S}$

$G \times V(\Gamma_{G, S}) \rightarrow V(\Gamma_{G, S})$

$g \cdot h = gh.$

This rule also tells you where edges go.

Also: $G \hookrightarrow \text{Sym}^+(\Gamma_{G, S})$
Thm. The natural map \( \Phi : G \to \text{Sym}^+(\Gamma_G, s) \) defined above is an isomorphism.

If we want to show surjectivity,
Let \( g \in \text{Sym}^+(\Gamma_G, s) \)
Need: \( \exists \sigma = \Phi(g) \) some \( g \in G \).
Which \( g \)? Take \( "g = \sigma(e)" \)
So \( \alpha \) & \( \Phi(g) \) agree on
the vertex \( e \).

Induct on distance from \( e \):
we'll show \( \Phi(g) \) & \( \alpha \) agree on
all vertices of distance \( n \)
from \( e \).

Base case: distance 0
Inductive step: Assume \( \Phi(g) \), \( \alpha \)
agree on vertices of distance \( n \)
from \( e \). Say \( v \) has distance \( n+1 \)
Then: \( \alpha(v) = \Phi(g)(v) \)
\( \alpha(w) = \Phi(g)(w) \)
\( s \) distance \( n \) from \( e \).
\( s \) or \( w \).

\( v \) or \( w \).
From Meier:

G ⊆ X = top space.

(Example: \( \mathbb{Z}^2 \subseteq \mathbb{R}^2 \))

A fundamental domain for the action is a subset \( F \subseteq X \) s.t.:

1. \( F \) closed
2. \( \bigcup_{g \in G} g \cdot F = X \)
3. Connected

and no proper subset of \( F \) satisfies 1 & 2.

In the example can take \( F = \) unit square

Issue #1. We don't know what closed subsets of a graph are.

Lots of fundamental domains:
For us: $G \leq \Gamma = \text{graph}$. 

$\Gamma' = \text{(barycentric) subdivision of } \Gamma$ (subdivide all edges of $\Gamma$).

A fundamental domain for $G \leq \Gamma$ is a subgraph $F \leq \Gamma'$ s.t.

1. $F$ connected.
2. $\bigcup_{g \in G} F = \Gamma'$

& $F$ minimal with respect to 1 & 2.

Examples:

1. $\mathbb{Z}/5 \mathbb{Z} \leq \Gamma$
2. $D_5 \leq \Gamma$
3. $\mathbb{Z}^2 \leq \Gamma$
Thm. Say $G \leq \Gamma$, connected
& $F \subseteq \Gamma'$ subgraph
& $Ug \cdot F = \Gamma'(\Gamma')$
\[g \in G\]
(e.g. $F =$ fund. domain)
Let $S = \{g \in G : g \cdot FnF \neq \emptyset\}$

Then $S$ generates $G$.

The smaller $F$ is, the smaller $S$ is.

That's why we care about fundamental domains.

Example. In $C^n$ graph

\[S = \{s, t\}\]

Q. Is $G$ acts faithfully
and $F$ is a fund domain, is $S$ minimal?
(Tolson).

Example next time:
Symn $G \uparrow Kn$. 
Thm. Say \( G \subseteq \Gamma \) connected & \( F \subseteq \Gamma \) subgraph & \( Ug \cdot F = \Gamma(\Gamma') \) \( g \in G \) (e.g. \( F = \text{fund. domain} \))

Let \( S = \{ g \in G : g \cdot F \cap F \neq \emptyset \} \)

Then \( S \) generates \( G \).

Proof. Let \( g \in G \)

Choose a vertex \( v \) in \( F \).

Find a path \( p \) from \( v \) to \( g \cdot v \) \((\Gamma \text{ connected})\)

Choose \( g_0, \ldots, g_n \in G \)

s.t. \( g_0 = e \), \( g_n = g \)

\[ U g_i \cdot F \text{ contains } p. \]

\[ g_i \cdot F \cap g_{i+1} \cdot F \neq \emptyset \quad \forall i. \]

Now: \( g_0 \cdot F \cap g_1 \cdot F \neq \emptyset \Rightarrow g_1 \in S \)

\[ g_1 \cdot F \cap g_2 \cdot F \neq \emptyset \]

\[ g_2 \in S \]

\[ g_2 \text{ is a product of two elts of } S. \]
Announcements Feb 4

- Cameras on
- Grade/topic due Fri Gradescope
- HW 3 due Feb 11 3:30
- Abstracts due Feb 26
- Office hours Fri 2-3, Tue 11-12, appt.

Today: Generators from group actions. \( SL_2 \mathbb{Z} \)
Fundamental domains - existence,...
Theorem \( G \cong \Gamma \) connected.
\( F \subseteq \Gamma' \) subgraph.

\[ U_{g \in G} g \cdot F = \Gamma' \]

Then
\[ S = \{ g \in G : g \cdot F \cap F \neq \emptyset \} \]
generates \( G \).

Example: \( \mathbb{Z} \setminus \{0, 1, 2, 3, \ldots \} \)
\( F = \emptyset \)
\( S = \{ \pm 1, 0 \} \)

Proof: Let \( g \in G \). Pick \( v \) = vertex of \( F \).

Choose a path \( p \) from \( v \) to \( g \cdot v \)
(\( \Gamma' \) connected)

Choose \( g_0 \cdot F, \ldots, g_n \cdot F \)
s.t. \( g_0 = e, g_n = g \), \( p \in \bigcup_{i=1}^{n} g_i \cdot F \)
\( g_i \cdot F \cap g_{i+1} \cdot F \neq \emptyset \).

Show by induction: \( g_i \) is a prod.
of elts of \( S_{\pm}^{\pm 1} \)
\( i = 0 \)

Assume true for \( i \). WTS for \( i+1 \).

\[ g_i \cdot F \cap g_{i+1} \cdot F \neq \emptyset \]
\[ g_i \cdot g_{i+1} \cdot F \cap F \neq \emptyset \]
\[ g_i \cdot g_{i+1} = s \in S \]
\[ g_{i+1} = g_i s \]
Example 1. $S_n \supseteq K_n$

$F = \text{half-edge from } n \text{ to } n-1.$

$S$ contains: $\text{Stab}(n) = S_{n-1}$

$(n-1 \ n)$ \text{ any elt of } S_{n-2}

Can simplify: $S_{n-1} \leftarrow \text{by induction; gen by adjacent transpositions.}$

$(n-1 \ n)$

$\Rightarrow S_n \text{ gen by adjacent transp.}$
Example 2. \( \text{SL}_2 \mathbb{Z} \cong \text{Farey graph} \).

Note: \( \exists A \in \text{SL}_2 \mathbb{Z} \) s.t. \( A \cdot (0) = (q) \) must be \(-1\)

\[
A = \begin{pmatrix} 0 & * \\ 1 & * \end{pmatrix}
\]

Better: \( \exists A \in \text{SL}_2 \mathbb{Z} \) s.t. \( A \cdot (0) = (q) \)

\[
A = \begin{pmatrix} p & * \\ q & * \end{pmatrix}
\]

Bezout

\[
* p + * q = 1
\]

\( \Rightarrow \) \( F \) need only 1 vertex of \( \Gamma \).

Note: \( (0,0) \) flips vertical edge.

Also need: \( \text{Stab} \ (0) \).

Vertices: \{ primitive \ \mathbb{Z} \ vectors \} / \pm

Edges: \( \begin{pmatrix} p \\ q \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) \iff \( \det \left( \begin{pmatrix} p & 0 \\ q & 1 \end{pmatrix} \right) = \pm 1 \).
What about \( \text{Stab}(i) \)?

\[
\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}
\]

only need: \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)

and: \( \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \) -I

(because first col is really \( \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \))

Finally:

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

generates \( \text{SL}_2\mathbb{Z} \).
Example $\mathfrak{X}_{3}$. $\text{SL}_2 \mathbb{Z}$ acts on the Farey graph.

$\text{GFarey graph}$

What is $F$?  

$\text{gen set}$

$(0 \ 1) \text{ takes } F \text{ to } F'$

$(0 \ 1) \text{ takes } F \text{ to } F'' \text{ & } F'''$

$(1 \ 0) \text{ fixes } F \text{ (and the whole tree)}$

$\frac{1}{2} \ 3$

These gems have finite order: $4, 6, 2$.  

$\pm (0)$
A more far out example.

Take $S = \text{surface}$

$$MCG(S) = \text{Homeo}(S)/\text{homotopy}.$$ 

We find generators using curve graph

vertices: simple closed curves in $S$ / homotopy
edges: disjointness
Announcements Feb 9

- Cameras on
- Abstract Feb 26 (consult w/me)
- HW 3 due Thu 3:30
- OH Fri 2-3, appt

Today: Fundamental domains
d_

Fundamental domains

Have \( G \Gamma \Gamma' \)

\( F = \) minimal, connected subgraph of \( \Gamma' \) so \( U_g F = \Gamma' \)

Lemma. If \( G \Gamma \Gamma' \) is connected graph then the action has a fundamental domain \( F \).

**Proof.** First assume \( G \Gamma \Gamma' \) has finitely many orbits of edges.

Color each edge according to orbit.

Example: \( D_5 \times \Gamma \) 10-gon has 1 orbit of edges

\( Z/5 \times \Gamma \) 10-gon has 2 orbits of edges

Build \( F \) inductively.

Choose any edge. Call it \( F \).

Find a new color edge (not in \( F \)), adjacent to \( F \), and add it to \( F \).

This stops by \( \circledast \)
Need to show $U_g.F = \Gamma'$. We should have added $g^{-1}e$ already.

Suppose not. There is an edge $e$ not in $U_g.F$ and adjacent to it. Say $e$ adjacent to $gF$. Then $g^{-1}e$ is adjacent to $F$.

This is a contradiction. We proved: $F$ is a union of edges from different orbits. So: any two translates of $F$ can only meet at vertices. So: the $\{g.F\}$ “tile” $\Gamma'$. 
Aside: There is a higher dimensional version.

$\text{SL}_2 \mathbb{Z}$

$\mathbb{Z}^2$

Fundamental domain

2D cell complex.
Thm. Say $G \triangleleft \Gamma = \text{conn. graph}$

$H \leq G$

$F_G \leq \Gamma'$ fund. dom for $G$

$F_H \leq \Gamma'$ fund dom for $H$

If $F_H = g, F_G \cup \cdots \cup g \cap F_G$

then $[G:H] = n$.

Examples

1. $7\mathbb{Z} \leq D_5$

2. $n\mathbb{Z} \leq \mathbb{Z}$

$n = 2$
Thm. Say $G \leq \Gamma = \text{conn. graph}$

$H \leq G$

$F_H \subseteq \Gamma'$ fund. dom for $H$

If $F_H = g_1 F_G \cup \ldots \cup g_n F_G$

then $[G : H] = n$.

Pf. Define

$\{g_i : F_G\}_{i=1}^n \rightarrow \{G/H\}$

$g_i : F_G \mapsto g_i H$

Want: bijection

Injectivity

Suppose $g_i H = g_j H$

$\Rightarrow g_i g_j \in H$

$\Rightarrow g_i g_j$ does not identify two pts in interior of $F_H$

$\Rightarrow g_i : F_G = g_j : F_G$

otherwise $g_i g_j$ takes $g_j : F_G$

to $g_i : F_G$ → FINISH
An application

\[ SL_2(\mathbb{Z})[m] = \text{level } m \text{ congruence subgp of } SL_2(\mathbb{Z}) = \{ A \in SL_2(\mathbb{Z}) : A \equiv I \mod m \}. \]

In \( SL_2(\mathbb{Z})[2] \): \( \pm I, \begin{pmatrix} 3 & 0 \\ 2 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \)

\[ \text{index 6 in } SL_2(\mathbb{Z}) \]

\[ \text{index 12 in } SL_2(\mathbb{Z}) \]
Chap 2. Groups gen. by reflections
(Coxeter groups)

Infinite Dihedral $G_0 = \text{Sym}(\begin{array}{cccc}
-2 & -1 & 0 & 1 \\
& & 1 & 2 \\
& & & 3
\end{array}) = D_{\infty}$

Sample elts: translate by $n$
reflect in vertex
reflect about middle of edge.

It is gen by reflectors about 0, $\frac{1}{2}$.

More next time!
Announcements Feb 11

- Cameras on
- Abstracts Feb 26: consult with me
- HW4 due Thu 3:30
- Office hours Fri 2-3, Tue 11-12, appt

Today: Do

Triangle gps
Coxeter groups.
From last time:

**Thm.** If \( G \leq \Gamma \)

- \( \text{fund dom } F \) and \( g \cdot F = f \)

\[ H \leq G \quad \Rightarrow \quad g = \text{id.} \]

- \( \text{fund dom } F_H \)

\[ F_H = g_1 \cdot F \cup \ldots \cup g_n \cdot F \]

then \([G:H] = n\).  

index 2

- \( \mathbb{Z} \leq \mathbb{C} \)  

\( \mathbb{Z} \times 1 \leq \mathbb{Z} \times \mathbb{Z}/2 \) index 4

Noah's question:

Take \( G \leq \Gamma \)  

index n \( H \leq G \)  

\( F_H \)

Now: \( K \) other gp.

\( G \times K \leq \Gamma \)  

\( H \times 1 \leq G \times K \) index bigger.

Same fund domains as before?

If yes: seems like contradiction.

\( \text{Fix} \)
Infinite Dihedral Group

\[ 
\Gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ D_{\infty} = \text{Sym}(\Gamma) \]

Last time: gen. by
\[ a = \text{refl. about } 0 \]
\[ b = \text{refl. about } \frac{1}{2} \]

\[ \text{Presentation?} \]

To start: \[ a^2 = b^2 = \text{id} \]

Typical elt of \( g \phi: \]
\[ aba^{-1} b^7 a^7 b^4 \]
really, this is: \[ ababebab \sim a \]

alternating word in \( a, b \).

So all elts are:
\[ (ab)^n \quad (ab)^n a \]
\[ (ba)^n \quad (ba)^n b \quad n > 0 \]

Presentation: \[ D_{\infty} \cong \langle a, b \mid a^2 = b^2 = \text{id} \rangle \]
A subgp of $D_\infty$

$H = \langle a, bab \rangle = \text{subgp gen by } a, bab \text{ in } D_\infty.$

By the way:

$H = \text{kernel of }$ \\
$D_\infty \rightarrow \mathbb{Z}/2$ \\
"count # of b's mod 2"

An explicit $H \rightarrow D_\infty$

$a \mapsto a$ \\
bab $\mapsto b$

$[D_\infty : H] = 2.$
Triangle groups

$W_{333} = \text{gp gen. by reflections in}$

$rb$?

grg = reflection about image of $r$ under $g$.

gbg

$r_b = \text{rotation by } 2\pi/3$

Some relations: $r^2 = b^2 = g^2 = \text{id}$

$(rb)^3 = (rg)^3 = (gb)^3 = \text{id}$. 

Goals: Fund. domain, Presentation
Guess for fund domain: original triangle.

To this end... take tiling of $\mathbb{E}^2$ by $\triangle$

& color the edges:

Critical point: this coloring is well-defined.
Each color is tiling by regular hexagons.

Check: \(r,g,b\) present these 3 hexagonal tilings.
We just showed

Prop. The coloring is well defined.

Cor. If \( g \in W_{333} \) & \( g \cdot T_0 = T_0 \)
then \( g = \text{id} \).

So the fund domain is at least as big as \( T_0 \).

To show \( T_0 \) is a fund domain, need that \( W_{333} \) acts trans. on triangles. Equivalently \( W_{333} \cdot T_0 = \mathbb{H}^2 \).

Prop. Let \( T \) be a triangle of the tessellation.

and \( T_0, T_1, \ldots, T_n = T \)
is a seq of triangles s.t. \( T_i \cap T_{i+1} \) is an edge colored \( c_i \in \{r,g,b\} \).

Then \( c_1 \ldots c_n \cdot T_0 = T \).
Prop. Let $T$ be a triangle of the tesselation. and $T_0, T_1, \ldots, T_n = T$ is a seq of triangles s.t. $T_i \cap T_{i+1}$ is an edge colored $C_i \in \{r, g, b\}$. Then $C_i \ldots C_n \cdot T_0 = T$.

Pf. Induct on $n$.

$n = 0 \checkmark$

Inductive hyp:

$C_i \ldots C_{n-1} \cdot T_0 = T_{n-1}$

Define $T' : \hspace{1cm}$

Note $T' = C_n T_0$

Have $C_i \ldots C_{n-1} T' = T_n$

$C_i \ldots C_{n-1} \cdot C_n T_0 = T_n \Box$
Coxeter groups: all generators have order 2.

all other relations:

\[(ab)^n = \text{id}.\]

e.g. \(D_n\).
Figure 7 from Coxeter’s address to the Royal Society of Canada
Announcements Feb 19

- Cameras on.
- HW 4 due Thu 3:30
- Abstracts Feb 26: consult with me before Feb 26.
- Take home midterm Mar 4
- Fri office hours moved (requests?)
  Office hours Tue 11-12, appt.
21. Let \( H \) be the subset of \( \text{Sym}_\mathbb{Z} \) so that \( h \in H \) iff \( H \) is finite \( C \subseteq \mathbb{Z}, \ k \in \mathbb{Z} \) s.t. \( h(n) = n + k \) for \( n \in C \).

(a) Show that \( H \) is a subgroup of \( \text{Sym}_\mathbb{Z} \).

*identity:* We see \( e \in \text{Sym}_\mathbb{Z} \) belongs to \( H \) by taking \( C = \emptyset, \ k = 0 \).

*inverses:* Let \( h \in H \) with associated \( C, k \). Let \( C' = C + k, \ k' = -k \).

Then \( h', C', k' \) satisfy the required
Indeed: $|C'| = |C| < \infty$ and

$$n \notin C' \Rightarrow n - k \notin C \Rightarrow h(n - k) = n \Rightarrow h'(n) = n - k = n + k'.$$

Composition. Let $h_1, h_2 \in H$ with associated $C_1, k_1$ & $C_2, k_2$.
Let $C' = (C_1 - k_2) \cup C_2$, $k' = k_1 + k_2$.
Then $h_1, h_2, C', k'$ satisfy the required conditions since
\[ |C'| = |(C_1 - k_2) \cup C_2| \leq |C_1 - k_2| + |C_2| = |C_1| + |C_2| < \infty \]

and \( n \notin C' \)

\( \Rightarrow n \notin C_2 \) and \( n \notin C_1 - k_2 \)

\( \Rightarrow n \notin C_2 \) and \( n + k_2 \notin C_1 \)

\( \Rightarrow h_1(h_2(n)) = h_1(n + k_2) = n + k_1 + k_2 = n + k' \)

(b) Show that \( H \) is finitely generated.

Let \( s = (0, 1) \)

\( t = (-1, 0, 1, 2, \ldots) \)
We will show that $s, t$ generate $H$.

Claim \((i, i+1) = t^i s t^{-i-1}\) \hspace{1cm} Claim!

Pf of Claim.

Let $h \in H$ with associated $C, k$.

Note $t^{-k}h$ has associated $C' = C - k$, $k' = 0$.

So $t^{-k}h$ can be regarded as an element of $Sym_{C'} \subseteq Sym_{2n}$.
Chap 3  Groups acting on trees.

3.1 Free groups.

\[ F_2 = \langle x, y \mid \rangle \]

\[ S = \{ x, y \} \]

For free group with free gens set.

\[ \Gamma_{F_2, S} \]

Paths in Cayley graph.

Reduced words in \( x, y \).

Non-backtracking.

For free group: no loops in Cayley graph.

Or: relations among generators

\[ \leftrightarrow \text{circuits in Cayley graph}. \]

So \( F_2 \cong T_4 = \text{reg. 4 valent tree} \).
The action $F_2 \circ T_4$

What does $x$ do?

Floret
Goal: Let $x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Then $x, y$ generate a subgroup of $\text{SL}_2\mathbb{Z}$, denoted $\langle x, y \rangle$.

Thus, $\langle x, y \rangle \cong F_2$.

In other words, every freely reduced word in $x^{±1}, y^{±1}$ multiplies to a nontrivial matrix.

False if you replace the 2's with 1's.

Indeed...

$\langle (01), (10) \rangle$

$= \text{SL}_2\mathbb{Z}$.

proof: row reduction.

which is not free because... torsion.

Exercise. Free groups are torsion-free.
**Ping Pong Lemma**

Say $G \leq X = \text{set}$

$a, b \in G$

$x_a, x_b \subseteq X$

nonempty, disjoint

$a^k \cdot x_b \subseteq x_a \ \forall \ k \neq 0$

$b^k \cdot x_a \subseteq x_b \ \forall \ k \neq 0$

Then $\langle a, b \rangle \cong F_2$.

**Pf by example**

Q. Why is $ababa^2a^3 \neq \text{id}$?

A. For any $x \in x_b$

$ababa^2a^3 \cdot x$ in $x_a$ hence $\neq x$. 
Ping Pong Lemma

Say \( G \subseteq X = \text{set} \)
\( a, b \in G \)
\( X_a, X_b \subseteq X \)
nonempty, disjoint
\( a^k \cdot X_b \subseteq X_a \ \forall \ k \neq 0 \)
\( b^k \cdot X_a \subseteq X_b \ \forall \ k \neq 0 \)

Then \( \langle a, b \rangle \cong \mathbb{F}_2 \).

Application 1

\( G = \text{SL}_2(\mathbb{F}) \)
\( a = (\frac{1}{0} \frac{2}{1}) \)
\( b = (\frac{1}{2} \frac{0}{1}) \)
\( X = \mathbb{Z}^2 \)
\( X_a = \{(p, q) \in \mathbb{Z}^2 : |p| > |q|\} \)
\( X_b = \{(p, q) \in \mathbb{Z}^2 : |q| > |p|\} \)

Check: If \( (p, q) \in X_b \), \( k \neq 0 \) then
\( a^k \cdot (p, q) \in X_a \)

\[
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-9 & 9 \\
1 & 0
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 \\
3 & 0
\end{pmatrix}
\]
Check: if \((\rho) \in X_b, k \neq 0\) then
\[ a^k \cdot (\frac{\rho}{q}) \in X_a \]
\[ a^k \left(\frac{\rho}{q}\right) = \left(\begin{array}{c} 1 \ 2 \ k \\ 0 \ 1 \end{array}\right) \left(\begin{array}{c} \rho \\ q \end{array}\right) \]
\[ = \left(\begin{array}{c} 1 \ 2^k \\ 0 \ 1\end{array}\right) \left(\begin{array}{c} \rho \\ q \end{array}\right) \]
\[ = \left(\begin{array}{c} \rho + 2^k q \\ q \end{array}\right) \]
Application 2. \( \text{Homeo}(\mathbb{R}) \)

\[
\text{Homeo}(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R} : f \text{ contin with } (\text{contin}) \text{ inverse} \} 
\]

group op: fog

Poll. If \( f: \mathbb{R} \to \mathbb{R} \) continuous & bij
is \( f^{-1} \) automatically contin.

(Yes)

Invertible: horiz. & vert. line test.

Inverse: flip over \( y = x \).

Goal: \( \mathbb{F}_2 \leq \text{Homeo}(\mathbb{R}) \).
Let $f(x)$ be: \[ X_f = \frac{2}{15} \text{ nbd of } \mathbb{Z} \]

g(x) \text{ is same, shifted right by } \frac{1}{2} \text{ up by } \frac{1}{2}

g(x) = f(x-\frac{1}{2}) + \frac{1}{2}

Prop. \langle f(x), g(x) \rangle \cong F_2

Ping pong! $X = \mathbb{TR}$

\[ X_f = \bigcup_{n \in \mathbb{Z}} [n-\frac{1}{15}, n+\frac{1}{15}] \]

\[ X_h = \bigcup_{n \in \mathbb{Z}} [n-\frac{1}{15} + \frac{1}{2}, n+\frac{1}{15} + \frac{1}{2}] \]
Next time: $F_3 \leq F_2$ (and $F_2 \leq F_3$)

index 2.

$& F_{\infty} \leq F_2$

$\mathbb{Z} \leq \mathbb{Z}$

index $\infty$. 

index $n$. 

index 2.
Announcements Feb 18

- Cameras on
- HW 5 due Thu
- Abstracts Feb 26 consult with me ahead of time by meeting/chat/email
- Midterm Mar 4
- Fri office hour @ 10 (just tomorrow)
- Office hours Tue 11-12, appt.

Today: $F_3 \leq F_2$, $GCT$ freely $\Rightarrow G$ free
Ping pong lemma

Let $G$ be a set,$\ a, b \in G$.

Let $X_a, X_b \subseteq X$ be disjoint, nonempty sets such that $b^k(X_a) \subseteq X_b \quad \forall k \neq 0$ and $a^k(X_b) \subseteq X_a \quad \forall k \neq 0$.

Then $\langle a, b \rangle \cong F_2$.

If $g = a^* b^* a^* b^* a^*$ is a freely reduced word in $a, b$ then $g \neq \text{id}$.

If $g$ starts and ends in $b$.

If $g$ starts with $a$ and ends with $b$, for example $a^3 b^5 = g$ , then $g \neq \text{id}$.

Conjugate so starts and ends with $a$.

Then $a a^3 b^5 a^{-1} \neq \text{id} \Rightarrow g \neq \text{id}$. \hfill $\square$
3.2 $F_3 \leq F_2$

$F_2 = \langle x, y \rangle$

$H = \text{subset of } F_2 \text{ consisting of reduced words of even length.}$

Let $a = x^2$, $b = xy$, $c = xy^2$.

Thm. ① $H \leq F_2$ of index 2

② $H$ is gen by $a, b, c$.

③ $H \cong F_3$

Pf. ① Consider

$f_2 \to \mathbb{Z}/2$

$g \to \text{mod 2 word length}$

(or: $x \to 1 \text{ this defines } y \to 1 \text{ a homom}$)

② Write all words of length 2 in $\{x, y\}^\pm 1$ in terms of $a, b, c$.

e.g. $y^2 = c^{-1}b$ etc.

or use our thm... 1 edge + 6 half-edges
Let \( a = x^2, \ b = xy, \ c = xy \) then no nearby cancellations:

\[
\alpha_i \neq \beta_{i+1} \\
\beta_{i+1} \neq \alpha_{i+2} \\
\beta_{i-1} \neq \alpha_i^{-1}
\]

In other words: \( \alpha_i, \beta_{i+1} \) don't cancel.

Case by case check.

e.g. \( w_{i-1} \ w_i \ w_{i+1} \)

Set \( w_i = \alpha_i \beta_i \alpha_j \beta_i \in \langle x, y \rangle \) ?

We'll show: If \( w \) has a cancellation:

\[
\ldots \beta_{i-1} \alpha_i \beta_i \alpha_{i+1} \beta_{i+1} \alpha_{i+2} \ldots
\]

\( (5 \text{ choices} \left( x^{-1}, y \right) (xy)^{5 \text{ choices}}) \)
3.4 Free groups and actions on trees.

Say \( G \curvearrowright \Gamma = \text{graph} \).

The action is \textit{free} if

\[ g \cdot v = v \Rightarrow g = \text{id}. \]

\& \( g \cdot e = e \Rightarrow g = \text{id} \)

\( \forall g \in G, v \in V(\Gamma), e \in E(\Gamma) \)

\textbf{Example.} \( F_2 \curvearrowright T_4 \) free.

\( \mathbb{Z}/n \curvearrowright \text{n-cycle free} \)

\( Dn \curvearrowright \text{n-cycle not free} \)

\( G \curvearrowright \Gamma G, s \) free.

\textbf{Lemma.} Any action \( \mathbb{Z}/2 \) on a tree \( T \) is not free.

\textbf{Pf.} \( v \) = any vertex

\[ v \overset{\text{path in } T}{\longrightarrow} 1 \cdot v \]

\textit{Paths unique} \( \Rightarrow \mathbb{Z}/2 \) preserves the path.

\( \Rightarrow \mathbb{Z}/2 \) fixes the midpoint of path

\( \Rightarrow \) fixed edge or vertex. \( \Box \)

\textbf{Exercise:} generalize to \( \mathbb{Z}/m \).

\textbf{Cor.} If \( G \) has torsion (elt of finite order), then any \( G \curvearrowright \Gamma \) not free.
Poll: Is $\text{SL}_2(\mathbb{Z})[2] \subset \text{Farey tree}$ free?

- I fixes the whole tree.

No. -I has order 2...

Also can find a matrix that "rotates" any vertex

If an elt of $\text{SL}_2(\mathbb{Z})$

fixes an edge, it fixes both vertices:

$\text{Stab}(e) \subseteq \text{Stab}(v)$
**Thm.** If a group acts freely on a tree, it is free.

**Cor.** Subgroups of free gps a tree (hard to prove directly).

**Pf #1** of Thm

1. Ping Pong
2. \( G \cong T = \text{tree freely} \)
3. \( F = \text{fundamental dom.} \)
4. \( S = \{ g \in G : g \cdot F \cap F \neq \emptyset \} \)

Earlier theorem: \( S \) generates \( G \).

From \( S \) remove duplicates: \( X, X' \).

To show: group gen. by \( S \) (i.e. \( G \)) is free.

\( S = \{ s_1, s_2, \ldots \} \)
To check: $S_2 \cdot X \cdot C \cdot X_2$.

Apply $S_2$. 

$x^2$ 

$x$ 

$x$ 

$x$ 

$x_2$ 

$x_2$ 

$x_2$ 

$x_2$
Announcements Feb 23

- Cameras on
- HW5 due Thu
- Abstracts Fri Feb 26 Gradescope (Team submissions)
- Take home midterm March 4
- Office hours moved to 1:00 Thu
- Regular office hours Tue 11, appt
- Ask for help on HW!

Today

- Ping pong lemma
- Free actions on trees ↔ free groups
- Free actions on edges of trees ↔ free products
Ping Pong Lemma II

Lemma 3.10

Have $G \triangleleft X$ = set

$S \subseteq G$

$\forall s \in SS^{-1}: X_s \subseteq X \checkmark$

$0 \in X \setminus \bigcup_s X_s \checkmark$

and

$0 \in s \cdot X_s \forall s \in SS^{-1} \checkmark$

$2 \in s \cdot X_t \subseteq X_s \forall t \neq s^{-1}$

Then: $\langle S \rangle \cong F_s$

$\langle S \rangle$ means subgp gen by $S$.

$F_s$ = free gp on $S$.

Distinctions from P.P.L. I:

1. $X_s$'s not disjoint
   (replaced with existence of $p$)

2. Only need $s \cdot X_t \subseteq X_s \forall t \neq s^{-1}$
   (replaced with $s \cdot X_s \subseteq X_s$).

PF. Look where $p$ goes.
3.4 Free gps & actions on trees

**Thm.** If a group acts freely on a tree, then it is free.

**Pf #1** Say $G$ acts freely on $T$.

Let $F$ be fund dom.

Call the $g.F$'s tiles.

$S = \{ g \in G : g.F \cap F \neq \emptyset \}$

Previous thm $\Rightarrow S$ generates $G$.

To show: $S$ generates a free gp.

Ping pong!

$X = \{ \text{tiles} \}$

$X_s = \{ \text{tiles that lie in component of } T \setminus F \text{ containing } S.F \}$

Let $p = F$.

Freeness $\Rightarrow s_i \cdot F \cap F \neq F \forall i$

$\Rightarrow 0$ in PPL. II.

1 in PPL is by defn.

Remains to check $\bullet$. 
\( X = \{ \text{tiles} \} \)

\( X_s = \{ \text{tiles that lie in component of } T \setminus F \text{ containing } s \cdot F \} \)

Consider the seq. of adjacent tiles:

\( s_i^* : F \rightarrow F \rightarrow S_2 F \rightarrow \text{rest of } X_{S_2} \)

Apply \( s_i : \\
F \rightarrow s_i F \rightarrow s_i S_2 F \rightarrow s_i X_{S_2} \)

\( s_i \cdot X_{S_2} \subseteq X_{S_1} \)

since \( T \) is a tree!

Remains to check \( 2 \).
Tricky Special case: \( s_i \cdot X_s \subseteq X_s \)

\[
\begin{align*}
  s_i' \cdot F & \to F \to s_i F \to \text{rest of } X_s, \\
  \text{Apply } s_i: & \to F \to s_i F \to s_i S, F \to s_i \cdot X_s, \\
  \text{since } T \text{ is a tree!} & \subseteq X_s, \\
  \Rightarrow & \text{any gp with elt of order } 2 \\
  \text{does not act freely on a tree,} & \\
  s_i v & \subseteq X_v.
\end{align*}
\]

Fine if \( s_i s_i F \neq F \)

\( \text{i.e. } s_i^2 = \text{id}. \)

Lemma from last time:

\( \exists/2 \) does not act freely on a tree.

Is this page needed???
Cor. Subgroups of free gps are free.

**Proof:**

- Let $F = \text{free gp}$
- $F \triangleleft T$ some tree (Cayley graph) freely.
- Any subgp inherits a free action.
- Apply the theorem.

Example $SL_2(\mathbb{Z})[m]$ is free $m \geq 3$.

Check freeness.

Matrices fixing center vertex:

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

etc.
Thm. If a group acts freely on a tree, then it is free.

Pr. 2 (ONGGT)

If \( G \) acts freely.

\( F = \text{fund dom} \)

\( \sim S = \{ s \in G : s \cdot F \cup F \neq \emptyset \} \).

Take a freely reduced word in \( S \).

\( w = s_1 \cdots s_k \quad s_i \in S \).

Want: \( w \neq \text{id} \).

Will show: \( w \cdot F \neq F \).

Will show \( F, s_1 \cdot F, s_1 s_2 \cdot F, \ldots, s_1 \cdots s_k \cdot F \) is a non-backtracking sequence of adjacent tiles.

Since \( T \) is a tree, this implies \( s_1 \cdots s_k \cdot F \neq F \).

Check \( s_1 \cdots s_i \cdot F \) adjacent, and not equal to \( (s_1 \cdots s_i) s_{i+1} \cdot F \).

\( s_{i+1} \cdot F \) adjacent to \( F \) (not equal \( F \) by freeness).

Apply \( s_1 \cdots s_i \) to both.

So: \( \{ \text{paths of tiles} \} \leftrightarrow \{ \text{freely reduced words} \} \).
Ping Pong Lemma II

**Lemma 3.10**

Have \( G \trianglelefteq X = \text{set} \)

\( S \subseteq G \)

\( \forall s \in S \subseteq X \times X \)

\( 0 \subseteq X \setminus \bigcup_{s} X_{s} \)

and \( 0 \subseteq X_{s} \forall s \in S \subseteq X \times X \)

Then: \( \langle S \rangle \cong \Gamma_{S} \)

After class, we decided that we need to assume \( S \) has no elts of order 2.

**Example:** \( \varphi_{3} = \{ 1/2 \} \cup \{ -1 \} \)

\( S = \{ -1 \} \)

\( p = +1 \)

\( X_{-1} = \{ -1 \} \)

\( 2 \) is vacuous here!

Proof of Thm is ok because we had a lemma about elts of order 2.

Noah suggested an alternate fix where we remove \( t \neq S^{-1} \). Not sure if this version has any application
Announcements Feb 25

- Cameras on
- Abstracts Fri Feb 26 Gradescope (Team submissions)
- HW 6 due Thu
- Take home midterm March 4
- No office hour Fri this week.
- Regular office hours Tue 11, appt
- Ask for help on HW!

Today
- Ping pong lemma
- Free actions on trees ⇔ free groups
- Free actions on edges of trees ⇔ free products
After class, we decided that we need to assume $S$ has no elts of order 2.

Example: $\mathbb{F}_{5}^{13} = 7412 \oplus 5 \oplus 13$

$s = 5 \oplus 13, p = +1 -1, p = -1$

$x_{-1} = \{13\}$

$\phi$ is vacuous here!

Pf of Thm is ok because we had a lemma about elts of order 2.

Noah suggested an alternate fix where we remove $t \neq S^{-1}$. Not sure if this version has any application.
3.4 Free gps acting on trees

Thm. If a group acts freely on a tree, it is free.

Pf. #2 Let $G \leq T = \text{tree freely.}$

$F = \text{fund. dom.}$

$\Rightarrow S = \{g : g \cdot F \cap F \neq \emptyset\}$

is a gen set $\Rightarrow \langle S \rangle = G$

Call the $g \cdot F$ tiles

The action preserves the tiling.

Want: no relations among $S$. Let $w = s_1 \ldots s_k$ word in $S/elt$ of $G$

Want $w \cdot F \neq F$ $(\Rightarrow w \neq \text{id})$ True because the word $w$ gives...

a non-backtracking path of tiles from $F$ to $w \cdot F$:

$F, s_1 \cdot F, s_1 s_2 \cdot F, \ldots, s_1 \ldots s_k \cdot F$

Indeed $s_1 \ldots s_{i+1} \cdot F \text{ adj to } s_i \ldots s_i \cdot F$ Apply $s_i \ldots s_i \cdot F$ to $F, s_i \ldots s_i \cdot F$
Examples

1. $\mathbb{Z} \subset \mathbb{C}$

2. $\text{Sym}^+ \left( \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right)$

3. $\text{SL}_2\mathbb{Z} \cap \mathbb{Z}$, Farey tree.
   
   Freely $m > 3$.

One vertex of Farey tree:

$\{ \pm(1), \pm(i) \}$

the elts of $\text{SL}_2\mathbb{Z}$ taking this vertex to itself:

$\pm I, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Sudipta Kolay.

of these, only $I \equiv 1 \pmod{m}$

$m > 3$. 
3.6 Free products of groups

A, B groups

\[ \mathbb{Z} \cup \mathbb{Z} = \mathbb{Z} \]
\[ \mathbb{Z} \uplus \mathbb{Z} = 2 \text{ copies of } \mathbb{Z}. \]

A word in \( A \uplus B \) is freely reduced if alternates between non-trivial elts of A & B, e.g.:

\[ a, b, a_1 b_2 a_3 \]

\[ A \ast B = \{ \text{freely red. words in } A \uplus B \} \]

group op: concat & reduce.

\[ (a, b_1)(b_2 a_2) = a_1 (b_1 b_2) a_2 \]

\[ (b_3 = b_1 b_2 \text{ in } B) \]

Prop. \( A \ast B \) is well-defined:

any word can be reduced to a unique freely red. word. Just like for \( F_n \).

Examples

\[ \mathbb{Z} \ast \mathbb{Z} \cong F_2 \]
\[ (5 \ 7 \ -3 \ 10)(-1 \ 1) \]
\[ = 5 \ 7 \ -3 \ 9 \ 1 \]
\[ (x^5 y^7 x^{-3} y^10)(y^{-1} x) \]
\[ = x^5 y^7 x^{-3} y^9 x \]
2. \( \mathbb{Z}/2 \times \mathbb{Z}/2 \cong D_\infty \)
   alternating words in 1, 1
   " " " " a, b
   \( \cong \langle a, b \mid a^2 = b^2 = \text{id} \rangle \)

3. \( \mathbb{Z}/2 \times \mathbb{Z}/3 \cong ? \)
   \( \cong \langle a, b \mid a^2 = b^3 = \text{id} \rangle \)

---

Some true things

1. \( A, B \leq A \times B \)
2. \( A \times B \rightarrow A \) (or B)
   kernel: \( B \)
3. \( A \times B \rightarrow A \times B \)
   kernel is free group (next time?)

E.g. \( D_\infty \rightarrow \mathbb{Z}/2 \)
   word length mod 2
   kernel: \( \mathbb{Z} \)
If \( G \cap T \) denote stabilizer of \( v \) by \( G_v \) freely and transitively. Then \( G \cong G_v \ast G_w \).

**Step 1.** \( G \) is gen by \( G_v \) & \( G_w \).

\[ S = \{ g : g \cdot F \cap F \neq \emptyset \} \]

\[ e \cdot v \]

\[ v \xrightarrow{g \cdot e} v' \]

\[ g \cdot e \in G_w. \quad \text{since } v, w \text{ in distinct orbits.} \]

**Step 2.** Take a freely red word in \( G_v \parallel G_w \)

\[ w = a_1 b_1 a_2 b_2 \ldots a_k b_k \]

To show \( w \neq \text{id} \) or \( w \cdot F \neq F \).

Like last time:

\( F, a, F, a_1 b_1 F, \ldots \) is a non-backtracking path.

E.g. \( F \xrightarrow{b_1 \cdot F} a_1 \xrightarrow{a_1 F / a_1 b_1 F} \)

□
Thm. If \( G \) \( \triangleleft \) \( T \) freely and transitively on edges and fundamental domain \( \mathcal{F} = v \cdot e \cdot w \) and \( v, w \) in distinct \( G \)-orbits. Then \( G \cong G_v \ast G_w \).

Application \( \text{SL}_2 \mathbb{Z} \) \( \cong \) Farey tree
\[
\text{PSL}_2 \mathbb{Z} = \frac{\text{SL}_2 \mathbb{Z}}{\pm 1} \\
\cong \mathbb{Z}/2 \ast \mathbb{Z}/3.
\]
Announcements Mar 2

- Cameras on
- HW 6 due Thu
- Midterm Mar 4-11
- Office hours Fri 2-3, Tue 11-12, appt.

Today: Free products & trees
      Free products are virtually free.
3.6 Free products

\[ A \ast B \]

Thm. \( G \cong T \) = tree.

freely, transitive on edges.

2 orbits of vertices

fundamental domain \( e \)

Then \( G \cong G_v \ast G_w \).

Pf. Step 1. \( S = \{ g \in G : g \cdot F \cap F \neq \emptyset \} \)
\[ = G_v \cup G_w \]

generates \( G \)

Step 2. Any word \( w = a, b, \ldots \)
\[ a \in G_v \quad b \in G_w \]

nonbacktracking

gives a path from \( e \) and \( w \).

the path is:
\[ e, a \cdot e, a, b \cdot e, \ldots \]

\[ \Rightarrow w \cdot e \neq e \]

\[ \Rightarrow w \neq id. \]

Application: \( \text{PSL}_2 \mathbb{Z} \cong \mathbb{Z}_2 \ast \mathbb{Z}_3 \)
Thm 3.28 Say $A \times B$ is a free prod. Then $T$ bipartite tree $T$ and an action of $A \times B$ satisfying the last theorem. If $|A|, |B| < \infty$ then $T = T(|A|, |B|)$.

Pf. Vertices: cosets of $A$ in $A \times B$.

Then $f$ edges: $g \rightarrow gA$, "g edge" $g \in A \times B$.

Action: left mult $gA hB$. $gh \in A$ or $B$.

Q. When do $g$- & $h$-edges intersect?

A. $gh \in A$ or $B$. 

Diagram: A tree with vertices labeled $A$ and $B$ and edges connecting them.
Pf. vertices: cosets of $A$
white vertices: cosets of $B$ in $A \times B$.

edges: $hA \rightarrow kg \rightarrow kgA \rightarrow kgB \rightarrow hB$;

edges: $hA \rightarrow kg \rightarrow kgA \rightarrow kgB \rightarrow hB$.

because can't have $gA = hB$.
If $gA = hB$ then $h^{-1}gA = B$ but $id \in B \Rightarrow id \in h^{-1}gA$
$\Rightarrow h^{-1}gA = A$. But $A \neq B$.

2) Action is free on edges. ✓
3) Two orbits of vertices (same as bipartiteness)
4) Transitively on edges. ✓

Check things:

1) $T$ is bipartite.
2) $A$ vertices
3) $T3$ vertices
5. $T$ is connected $\in \mathbb{A} \times \mathbb{B}$

To connect id-edge $e$ to g-edge:
write $g = a \cdot b$, ...
the path of edges is $e, a \cdot e, a \cdot b \cdot e, ...$

6. $T$ is acyclic.
Nonbacktracking paths $\leftrightarrow$ freely red. words
The stabilizer of the vertex \( A \) is \( A \)

Prop. The stabilizer of \( gA \) is \( gAg^{-1} \)

\[ gAg^{-1}gA = gA \]

Poll. Consider \( \langle a, bab^{-1} \rangle \leq F_2 \). Is it free?

Yes. Same as proof at start of class... A reduced word in \( a, c \) gives a non-back path (2 edges for each "syllable").
Let $A, B$ be finite groups. Then $A \times B$ is virtually free (it has a free subgroup of finite index).

We'll prove more: kernel $K$ of $A \times B \to A \times B$ is free. Kernel has index $|A \times B| < \infty$.

Make the tree $T$ for $A \times B$ as above.

Check $K$ acts freely.

Stabilizers of edges in $A \times B$, nontrivial hence $K$, are trivial.

Stabilizers of vertices in $A \times B$ are of form $g a g^{-1}$, $a \in A$ not $id$, which maps to $a \times id$ in $A \times B$.
HW #20

$\mathbb{Z}_2 \times \mathbb{Z}_3 \rightarrow A_4$

$K = \text{kernel.}$

Stabilizer of $A$ is $A$.
Now check $A \cap K = \{\text{id}\}$.
For each vertex, find stabilizer (by Prop), show intersection with $K$ is $\{\text{id}\}$. 
Generalizing to $A \times B \times C$.

$\mathbb{Z}_2 \times \mathbb{Z}_3$
ANNOUNCEMENTS Mar 9

- Cameras on
- Midterm due Thu 3/11 3:30
- First draft due Mar 26 Apr 2
- Office Hours Wed 11-12, Thu 10-10:50, appt

Today: Word problem
Normal forms
BS(1,2)

\[
\begin{align*}
(12) & \in \\
SL_2(\mathbb{Z})[2] & = \{ A \in SL_2 : A \equiv I \mod 2 \} \\
& = \ker (SL_2 \to SL_2(\mathbb{Z})) \\
& \text{(no OH Fri this week)}
\end{align*}
\]
5  Word problem

Given \( G = \langle S \mid R \rangle \)

\( \{S\}^* \) = words in \( S \\ S^{-1} \)

\( \pi : \{S\}^* \rightarrow G \)

Word Problem (Dehn): Determine if a given \( w \in \{S\}^* \)

has \( \pi(w) = \text{id} \).

We say WP is solvable if there is an algorithm...

Ball of radius \( n \)
in \( G_3 \): \text{union of paths from \text{id} of length } \leq n

Equivalent to WP:

1. Equality problem (does \( \pi(w_1) = \pi(w_2) \))
   (same as: \( \pi(w_1 w_2^{-1}) = \text{id} ? \))

2. determine which paths in Cayley graph are loops.

2') F algorithm to draw ball of radius \( n \) in the Cayley graph.

First example: \( G = \langle a,b \mid ab=ba \rangle \cong \mathbb{Z} \times \mathbb{Z} \)

solution to WP: exponent sum.

Second example: \( G = \langle a,b \rangle = F_2 \)

solution to WP: freely red.

\( BS(m,n) \) harder... (later today)
A simple example of a group with unsolvable word problem

Donald J. Collins

Generators:

\[ a, b, c, d, e, p, q, r, t, k. \]

Relations:

\[
\begin{align*}
p^{10}a &= ap, & p^{10}b &= bp, & p^{10}c &= cp, & p^{10}d &= dp, & p^{10}e &= ep, \\
q^a &= aq^{10}, & q^b &= bq^{10}, & q^c &= cq^{10}, & q^d &= dq^{10}, & q^e &= eq^{10}, \\
ra &= ar, & rb &= br, & rc &= cr, & rd &= dr, & re &= er, \\
pacqr &= rpcaq, & p^2adq^2r &= rp^2daq^2, \\
p^{3}bcq^2r &= rp^{3}cbq^2, & p^{4}bdq^4r &= rp^{4}dbq^4, \\
p^{5}ceq^5r &= rp^{5}ecaq^5, & p^{6}deq^6r &= rp^{6}edboq^6, \\
p^{7}cdeq^7r &= p^{7}cdeeq^7, \\
p^{8}caaq^8r &= rp^{8}caaq^8, \\
p^{9}daaq^9r &= rp^{9}daaq^9, \\
p^t &= tp, & q^t &= tq, \\
k^{-1}(aaa) = k^{-1}(aaa) & \quad (\text{for } k \neq 1) 
\end{align*}
\]
How can WP be hard?

$\mathbb{Z}^2$ example

Relations: pushing across squares.

Given a word $w$ with $\pi(w) = \text{id}$, can make it monotonically shorter using relations.

To have unsolvable WP must be that short words need many relations (which make the word much longer before getting shorter).

Dehn functions (ONGGT)
Word Problem for $BS(1,2)$

$BS(1,2) = \langle a,t \mid tat^{-1} = a^2 \rangle$

Let $G = \{ \text{linear fn}s \, g: \mathbb{R} \to \mathbb{R} \}$

of form $g(x) = 2^x + \alpha$

with $\alpha \in \mathbb{Z}[1/2]$?

Check: $G$ is a group.

Have $f: BS(1,2) \to G$

$a \mapsto g(x) = x + 1$

$t \mapsto g(x) = 2x$.

Prop. $f$ is an isomorphism.

Cor. $f$ has solvable WP (evaluate $f(w)$).

Pf. Last time: well-def. $f(tat^{'}) = f(a^2)$

Suri. $f(t^{-k}a^m t^k) = \left( g(w) = x + \frac{m}{2^k} \right)$

\[ f(a^n) = (g(x) = 2^n x) \]

Inj. Say $f(w) = \text{id}$.

key: exponent sum on $t$'s is 0.

(take derivative, chain rule)

So: if there are $t$'s, there are $t$'s.

Can conjugate so have $t \frac{a^k}{a^1} t^{-1}$.

Replace with $a^{2k}$.

Eventually $a^n \Rightarrow n=0 \quad \square$
Example

This shows:

If exp. sum on $t$ is 0 then $w \sim a^n$
Cayley graph for $BS(1,2)$

Poll: shortest path to $a^{33}$

Two path of length 11.
Announcements Mar 11

- Cameras on
- Midterm due 11:59 pm
- No HW this week
- First draft due Apr 2
- Office hours by appt.

Today

Normal forms...
  in BS(1,2)
  in B_3
Hyperbolic plane?
Normal Forms

\[ G = \text{group} \]
\[ S = \text{gen set} \]

We have

\[ \pi: \{\text{words in } SU^{-1}\} \rightarrow G \]

A normal form for \( G \) is an

\[ \eta: G \rightarrow \{\text{words in } SU^{-1}\} \]

s.t. \( \pi \circ \eta = \text{id} \).

To tell if two elements are same, put them in normal form & compare.

This solves word problem.

We can also think of a normal form as a subset of \( \{\text{words in } SU^{-1}\} \), one word in \( \pi^{-1}(g) \) for each \( g \in G \).

Examples. ① \( \mathbb{Z}^2 = \langle a, b | ab = ba \rangle \)

normal form: \( \{a^m b^n : m, n \in \mathbb{Z}\} \)

② \( \mathbb{F}_2 \) normal form: freely reduced words
Normal forms for $BS(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$

We know

$$BS(1,2) \xrightarrow{\sim} \{ g(x) = 2^n x + \frac{m}{2^k} : m, n, k \in \mathbb{Z} \}$$

\[ a \mapsto g(x) = x + 1 \]
\[ t \mapsto g(x) = 2x \]

We can check:

\[ t^{-k} a^m t^k t^n \mapsto g(x) = 2^n x + \frac{m}{2^k} \]

Guess for normal form:

\[ \{ t^{-k} a^m t^k t^n \} \]

\[ F A I L S ! \quad tat^{-1} = a^2 \]

\[ \begin{align*}
   & k = -1 \\
   & m = 1 \\
   & n = 0
\end{align*} \quad \begin{align*}
   & k = n = 0 \\
   & m = 2
\end{align*}

\[ U \{ t^n : n \in \mathbb{Z} \} \]

The ambiguity is that $m/2^k$ might be reduced, i.e. $m$ even.

The fix: write elements of $BS(1,2)$ as

or $g(x) = 2^n x$

$\mapsto$ Normal form:

\[ \{ t^{-k} a^{2m+1} t^{k+n} : k, m, n \in \mathbb{Z} \} \]
Normal form for $B_3$ (or $B_n$)

Generators:

\[ \sigma_1, \quad \sigma_2 \]

Multiplication is stacking.

\[ \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 = \triangle \]

Poll. Which are equiv to

\[ \begin{align*}
1 & 2 & 1 & 1 & 2 & 2 & 2 \\
2 & 1 & 2 & 1 & 2 & 2 & 2 \\
2 & 1 & 1 & 2 & 1 & 2 & 2 \\
2 & 1 & 1 & 1 & 2 & 1 & 2 \\
2 & 1 & 1 & 1 & 1 & 2 & 1
\end{align*} \]
Garside Normal Form

Ingredient # 1:

\[ B_3 \rightarrow Z \]
\[ \sigma_1 \rightarrow 1 \]
\[ \sigma_2 \rightarrow 1 \]

"signed word length"

Ingredient # 2:

\[ \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 = \Delta \]

"half-twist"

Running example: \( \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \)

Step 1. Replace each \( \sigma_i \) with \( \Delta \) pos. word.

Why can we do this?

\[ \Delta = \sigma_2 \sigma_1 \sigma_2 \]
\[ \Delta' \sigma_2 \sigma_1 = \sigma_2 \]
\[ \Delta' \sigma_1 \sigma_2 = \sigma_1 \]

example. \( \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \rightarrow \sigma_1 \sigma_2 \sigma_1 \Delta' \sigma_2 \sigma_1 \)

Step 2. Move all \( \Delta' \) to the left.

Why can we do this? \( \sigma_i \Delta' = \Delta' \sigma_{n-i} \)

example. \( \rightarrow \Delta' \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_1 \)
We now have $\Delta^i \cdot \text{pos word } i \leq 0$.

Step 3. Find maximal $i$ so our braid is $\Delta^i \cdot \text{pos word } i \leq 0$.

In our example:

$$\Delta^1 \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_1 = \Delta^1 \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_1 \sigma_2 \sigma_1$$

How do we know our example is not $\Delta^0 \cdot \text{pos word } ?$

It is! $\Delta^0 \sigma_2 \sigma_1$

In general, use ingredient #1.
Alternate example: (not related to running example)

How do I know \( \Delta^{-1} \sigma_i \neq \Delta^0 \cdot \text{pos word?} \)

\[ \downarrow \quad \downarrow \]

signed word length -2

signed word length \( \leq 0 \).

Another example:

How do I know \( \Delta^{-1} \sigma_i \Delta^0 \neq \Delta^0 \cdot \text{pos word?} \)

signed word length 1 \( \frac{\text{must have length}}{1} \)

only 2 such words.
Step 4: Find all $\Delta^i$ pos word equaling $g$.
Choose the smallest in lexicographic order.

Example: $\Delta^0 \sigma_2 \sigma_1$ normal form.
only candidates are $\Delta^0 \sigma_1 \sigma_2$, $\Delta^0 \sigma_1^2$, $\Delta^0 \sigma_2$, $\Delta^0 \sigma_2^2$.

Steps 3 & 4 use:

Thm. If two positive braids are equal, they differ by finitely many
inverses needed!

In fancy language:
The braid monoid $B^+_n$ embeds into $B_n$. 
Prove this theorem?

21221
12121
11211

12212

positive crossing

neg. crossing

\( N \)

\( K \)

\( \mu \)
Announcements Mar 18

- Cameras on
- First draft due Apr 2
- HW due Thu 3:30
- Office Hours Fri 2-3, Tue 11-12, appt
- Talk to me about extra credit.

\[ \langle (1^2), (1^3) \rangle \]

“lantern relation”

Question: What are all groups \( G \) of order \( n \) with \( [G,G] = A_n \)?

Hope: \( G = S_n \).

Today

Burnside problem
**Burnside Problem**

A group is a **torsion group** if all elements have finite order.

Finite groups are all torsion groups.

Easy to make infinite torsion groups:

\[ \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \cdots \]

\[ \mathbb{Q}/\mathbb{Z} \]

---

These are not finitely generated (why?)

Q. (Burnside 1902) Is there a fin. gen. infinite torsion group?

A. (Golod - Shafarevich '60s)

Yes.

We'll show an example from 80's by Gupta-Sidki using GGT.
Starting pt is...

$T = \text{rooted ternary tree}$

$V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow \ldots$

$\text{Sym}(T) = \text{root-preserving symmetries.}$

Important thing:
$\text{Sym}(T) \times \text{Sym}(T) \times \text{Sym}(T) \leq \text{Sym}(T)$

$\text{Sym}(T)$ is self-similar.
Two elements of $\text{Sym}(T)$

$\sigma : V_{n_1, \ldots, n_k} \rightarrow V_{(n_1+1)n_2, \ldots, n_k}$

$\omega = \sigma_0 \delta_1 \cdot \sigma_2 \delta_0 \cdot \sigma_2^{20} \delta_2^{21} \cdot \ldots$

$\omega_L$ means do $\omega$ at $V_L$.

Let $U = \langle \sigma, \omega \rangle \leq \text{Sym}(T)$
Let $U = \langle \sigma, \omega \rangle \leq \text{Sym}(T)$

then $U$ is a fin. gen. $\infty$ torsion group.

Need a "normal form" for $U$.

Then will do 1 & 2

\[\text{fin gen} \checkmark\]

1. $\infty$

2. torsion
A "normal form" for $U$

**Lemma 1.** Each elt of $U$ can be expressed as

$$\sigma^k x_1 \cdots x_n$$

where $x_i \in \{\omega, \sigma \omega \sigma^{-1}, \sigma^2 \omega \sigma^{-2}\}$

(kind of like $B_n$ normal form).

**If.** Need a relation

$$\sigma = \sigma^{-2} \Rightarrow \omega \sigma = \omega \sigma^{-2}$$

$$\Rightarrow \omega \sigma = \sigma(\sigma^2 \omega \sigma^{-2})$$

**Use the relation to push $\sigma$'s to the left.**

**Example**

$$\sigma \omega \sigma \omega$$

$$\sigma(\sigma^2 \omega \sigma^{-2}) \sigma \omega$$

$$\sigma^2 \sigma^2 (\sigma^2 \omega \sigma^2) \sigma^{-2} \omega$$

$$\sigma^2 (\sigma \omega \sigma^{-1})(\omega)$$

$$x_1 \quad x_2$$
Prop. \( |U| = \infty \).

We will find \( K \neq U \) and \( K \rightarrow U \).
The Prop follows.

Defining \( K \)

have \( U \rightarrow \mathbb{Z}/3 \)

action on three edges from root. Works because \( w \& \sigma \) preserves the cyclic order.

\( K \) is the kernel.

\( \ast \) In terms of "normal form" these are the \( \sigma^k x_1 \cdots x_n \) with \( k=0 \).

Let \( H = \langle \omega, \sigma \omega \sigma^{-1}, \sigma^2 \omega \sigma^{-2} \rangle \)

Lemma. \( K = H \).

Pf. Step 1. \( H \leq K \) \( \checkmark \)

Step 2. \( H \leq U \).

Finite check. Conjugate each gen for \( H \) by gen for \( U \), end up backin' \( H \)

Step 3. \( U/H \cong \mathbb{Z}/3 \) by Lemma 1
Lemma $K \rightarrow U$

If $K$ maps to the copy of $\text{Sym}(T)$ below vertex 0.

Want: Image of $K$ contains $\sigma_0$ & $\omega_0$

Check on generators:

$\omega \mapsto \sigma_0$
$\sigma_0 \omega \sigma_0^{-1} \mapsto \omega_0$
$\sigma^2 \omega \sigma^{-2} \mapsto \sigma_0^{-1}$

(2) Prep. $U$ is torsion: each elt has order a power of 3.

PF. Induction on syllable length in normal form,

$\sigma^k x_1 \cdots x_n$

$\sigma^k$ is a syllable, $x_i$ is a syllable.

Idea. Given $g$, show $g^3$ is a product of 3 commuting elements of shorter syllable length.
proof by example

\[ g = \sigma \omega \text{ syll. length } 2. \]

\[ g^3 = \sigma \omega \sigma \omega \]

normal form has \( k = 0 \):

\[ \sigma^3 (\sigma^{-2} \omega \sigma^2) (\sigma^{-1} \omega \sigma) \omega \]

lie in 3 different factors

\[ \text{Sym}(T) \times \text{Sym}(T) \times \text{Sym}(T) \]

all three pieces have syllable length 1.
Announcements Mar 23

- Cameras on
- HW due Thu 3:30
- Office Hours Fri 2-3, appt
  + makeup 10-11 Wed
- Progress report Apr 2 ~ 1 page
- First draft Apr 9
- Talk to me about makeup points!

Today
- Howson's thm
- Regular languages
- Automata
Howson's TNN

Thm 7.32 (1954)
If $G,H$ f.g. subgps of $F_n$ then $G\cap H$ is f.g.

A "counterexample" with $F_n$ replaced by another group:

Take $F_2 \times \mathbb{Z}$, $F_2 = \langle x,y \rangle$, $\mathbb{Z} = \langle z \rangle$

$G = F_2$ (first factor)

$H = \ker(F_2 \times \mathbb{Z} \to \mathbb{Z})$

all 3 gens $\mapsto 1$

To check:
1. $G$ f.g. ✓
2. $H$ f.g.
3. $G\cap H$ not f.g.

2. Claim: $H$ is gen by $\{xz^{-1}, yz^{-1} \}$

Step 1. $\langle S \rangle$ normal.
To show: $gSg^{-1} \subseteq \langle S \rangle$

$g = (\text{gen for } F_2 \times \mathbb{Z})^3, s \in S.$

example. $y(x^{-1}z)y^{-1} = (y^{-1})(x^{-1}z)(y^{-1}z)$

Step 2. $\langle S \rangle \subseteq H$ ✓

Step 3. $(F_2 \times \mathbb{Z})/\langle S \rangle \cong \mathbb{Z}$

We get $F_2 \times \mathbb{Z}$ subject to $x = z, y = z$
\[ F_2 \times \mathbb{Z} \quad F_2 = \langle x, y \rangle \quad \mathbb{Z} = \langle z \rangle \]

\[ G = F_2 \quad \text{(first factor)} \]

\[ H = \ker (F_2 \times \mathbb{Z} \to \mathbb{Z}) \]

all 3 gens \( \rightarrow 1 \)

Remains.

3. \( G \cap H \) not \( F_2 \).

\( G \cap H \) is the subgp of \( F_2 \):

\[ \ker F_2 \to \mathbb{Z} \]

\[ x, y \mapsto 1 \]

(exponent sum 0).

Claim. \( G \cap H \) is freely gen by

\[ x^i y^{-i} \]

Example. \( x^5 y^{-5} x^3 y^{-3} y^2 x^{-2} \)

Very similar to HW problem:

\[ \ker F_2 \to \mathbb{Z}^2 \]

\[ x \mapsto (1, 0) \]

\[ y \mapsto (0, 1) \]

Freely gen by

\[ \{ x^i y^j x^{-i} y^{-j} \} \]
Hanna Neumann Conjecture (1957)
\[ \text{rk (G \cap H)} - 1 \leq (\text{rk (G)} - 1)(\text{rk (H)} - 1) \]
for \( G, H \leq F_n \)

Proved in 2011 by Friedman, Mineyev.

Our proof of Howson's thm
uses regular languages, automata.

Today: automaton version of Howson's thm. Thu: Howson’s thm.
Languages

\[ S = \{ x_1, \ldots, x_n \} \text{ "alphabet"} \]
\[ S^* = \{ \text{words of finite length in } S \} \]

Any subset \( L \subseteq S^* \) is called a language.

Examples

1. \( S = \{ a, \ldots, z \} \quad L = \{ \text{words in OED} \} \)
2. \( S = \{ a \} \quad L = \{ a^n : 3 \mid n \} \)
3. \( S = \{ a, b, c \} \quad L = \{ a^i b^j c^k : i > 0, j \geq 0, k \geq 0 \} \)
4. \( S = \{ \text{gen set for } G \}^{\geq 1} \quad L = \{ \text{words in } S \text{ that equal id in } G \} \)
Automata (= simple computer)

$S$ = alphabet (finite set)

An automaton $M$ over $S$ consists of a directed graph with decorations:

- some subset of vertices called start states $\hat{S}$
- some subset $A$ of vertices called accept states $\hat{O}$
- edges labeled by elts of $S$.

If the graph is finite, $M$ is a finite state automaton.

The language accepted by $M$ is $\{w \in S^* : w$ given by a directed path in $M\}$

Examples

$\xrightarrow{a} \xrightarrow{a} \xrightarrow{a} \xrightarrow{a}$

$L = \{a^i : 3| i \}$
Poll: Is there a simpler automaton for the same language?

\( h = \{ \text{words with } b\text{-exponent even} \} \)

\( a^3 b^5 a b \checkmark \)
Deterministic automata

A det. aut. is a FSA with:
- exactly one start state
- no two edges leaving same vertex have same label
- no edges with empty label
  (in Meier: empty = \( \varepsilon \))

It is complete if each vertex has departing edges with all possible labels.

What's deterministic about it?

words \( \longleftrightarrow \) paths

\[
\text{\( w \)} \quad \xrightarrow{a} \quad \text{\( O \)} \quad \xrightarrow{a} \quad \text{\( a \)}
\]

The word \( wa \) corresponds to more than 1 path.

To see if a word is in the accepted language, start at the start state, trace out the word/path, see if land at accept state.

A language is regular if accepted by a det. FSA.
Automaton version of Howson's Thm

**Thm 7.11** Say \(K, L \subseteq \Sigma^*\) are regular languages. Then so are:

1. \(\Sigma^* \setminus K\)
2. \(K \cup L\)
3. \(K \cap L\)
4. \(KL = \{wkwL : wk \in K, wL \in L\}\)
5. \(L^* = L^{\infty}\)

**reg. lang. is automaton version of f.g.

**Lemma 1.** \(L\) accepted by a det. FSA

(i.e. \(L\) is regular) \(\Rightarrow\) \(L\) accepted by a complete det FSA.

**Proof.** (exercise: add dead ends/fail states)

**Lemma 2.** \(L\) accepted by a non-det. FSA \(\Rightarrow\) \(L\) accepted by a det. FSA.

In other words: starting with a non-det FSA, Lemma 2 converts it to a det FSA, Lemma 1 converts to a complete det FSA.
Lemma 2. \( L \) accepted by a non-det. FSA \( \Rightarrow L \) accepted by a det. FSA.

Two steps:
1. Get rid of arrows with empty labels
2. Get rid of multiple start states.

Example:
\[
\begin{align*}
\text{h} = \{ a^i b^j : i,j > 0 \} \\
\end{align*}
\]

New vertices: subsets of old vertices
New start vertex: set of all old start vertices
New accept vertices: all sets containing an accept
We can now convert $\mathcal{M}_\varepsilon$ into a deterministic automaton, $\mathcal{D}$. The states of $\mathcal{D}$ consist of all the subsets of $V(\mathcal{M}_\varepsilon)$. The single start state of $\mathcal{D}$ is the subset of $V(\mathcal{M}_\varepsilon)$ consisting of all the start states of $\mathcal{M}_\varepsilon$. The accept states of $\mathcal{D}$ are the subsets of $V(\mathcal{M}_\varepsilon)$ that contain at least one accept state of $\mathcal{M}_\varepsilon$. In $\mathcal{D}$ there is an edge from $U$ to $U'$ labelled by $x$ if, for each $v \in U$, there is an edge labelled $x$ from $v$ to some $v' \in U'$, and $U'$ is entirely composed of such vertices. That is, there is an edge labelled $x$ from $U$ to the vertex corresponding to the set

$$U' = \{v' \in V(\mathcal{M}_\varepsilon) \mid v' \text{ is at the end of an edge labelled } x \text{ that begins at some } v \in U\}.$$
is in vertex restriction to $V$ of $T_G$. Sym gives $\mu$.

If you look all $G$ and all $gT_G$ give a finite subset of $\text{Sym}^n$. Those are the finite States.
Announcements Mar 25

- Cameras on
- HW due Thu 3:30
- Office Hours Fri 2-3, Tue 11-12
- Progress report Apr 2 ~ 1 page
- First draft Apr 9
- Talk to me about makeup points!

Today
- Howson's thm
- Regular languages
- Automata
Howson's TNM

Thm 7.32 (1954)
If $G,H$ f.g. subgps of $F_{\infty}$ then $G \cap H$ is f.g.

Original proof: algebraic topology

Languages

$S = \{x_1, \ldots, x_n\}$ “alphabet”
$S^* = \{\text{words of finite length in } S\}$
Any subset $L \subseteq S^*$ is called a language

Examples

1. $L = \{a_i b_j : i,j > 0\} \subseteq \{a,b\}^*$
2. Consider $H = \langle a^2, b \rangle \leq F_2$
   
   $L = \text{reduced words in } a, b, a^{-1}, b^{-1}$
   corresponding to elts of $H$
   
   $\subseteq \{a, b, a', b'\}^*$
1. $L = \{a^i b^j : i, j > 0\} \subseteq \{a, b\}^*$

2. Consider $H = \langle a^2, b \rangle \leq F_2$
   $L$ = reduced words in $a, b, a^{-1}, b^{-1}$
   corresponding to elts of $H$.
   $\subseteq \{a, b, a^{-1}, b^{-1}\}^*$

Automaton examples

1. $a \xrightarrow{} 1 \xrightarrow{} b \xrightarrow{} 2$

2. Roughly states correspond to last letter used.
Deterministic FSA

FSA with
  - one start state
  - no edges w/ empty label
  - ≤ 1 edge with a given letter starting from each vertex

→ regular languages

Complete: = 1 in 3rd bullet.

Tidying up a FSA

Lemma 1. \( L \) accepted by det FSA

\[ \Rightarrow L \text{ accepted by complete det FSA} \]

Lemma 2. \( L \) acc by non-det FSA

\[ \Rightarrow L \text{ acc by det FSA}. \]

In other words: FSAs, det FSAs, compl. det FSAs all give same languages, i.e. regular lang's.
Lemma 2. \( L \) acc by non-det FSA
\[ \Rightarrow L \) acc by det FSA. \]

Pf. Given FSA \( M \), want to make it satisfy the 3 bullet pts without changing the accepted lan.

We'll just do 3rd bullet:

- \( \leq 1 \) edge with a given letter starting from each vertex

For each \( a \in S \) (= alphabet)

- Make an \( a \)-edge from \( U \) to \( V \)

\[ 1^{\text{st}} \text{bullet!} \]

\[ 3^{\text{rd}} \text{bullet} \]

- \( \{v_i : v_i \in V(M), \exists a \text{-edge from } v_i \text{ to } V \} \)

Start state \( \{ \text{start states in } M \} \)

Accept states elts of \( P(V(M)) \) cont. accepts

Let \( D \) be FSA with

Vertices \( V(D) = P(V(M)) \setminus \emptyset \)

Edges Let \( U = \{v_1, \ldots, v_k \} \in V(D) \)

\[ v_i \in V(M) \]

For each \( a \in S \)

- Make a \( a \)-edge from \( U \) to \( V \)

\[ \forall i = 1 \ldots k \}

Start state \( \{ \text{start states in } M \} \)

Accept states elts of \( P(V(M)) \) cont. accept.
Key part of defn of $D$:

Make an $a$-edge from $U$ to $V$ where

$$V = \bigcup_{i=1}^{k} \{ v \in V(M) : \exists a\text{-edge from } v_i \text{ to } v \}$$

$M$:

Can cut the chaff
Automaton version of Hewson's Thm

Thm 7.11 Say $K, L \subseteq S^*$ are reg. languages. Then so are:

1. $S^* \setminus K$
2. $KUL$
3. $KNL$
4. $KL = \{ wKwL : w \in K, wL \in L \}$
5. $L^* = LULLULLLU \ldots$

** reg. lang. is automaton version of f.g.

** Proof 
1. Toggle accept/non-accept states
   
   ![Diagram](image)

   example, $L = \{ a^i : i \text{ even} \}$

2. Say $M_k, M_l$ FSA for $K, L$
   then $M_k \cup M_l$ is a FSA for $KUL$. Apply the 2 lemmas

3. $KNL = S^* \setminus (S\setminus K) \cup (S \setminus L)$
   Apply 1 & 2
**Regular vs. finite gen.**

**Thm.** $S = \text{fin. gen. set for } G$

Then $H \leq G$ is fin. gen.

$\iff H$ is image of reg. lang.

$h \in (S^\pm)^*$.

under $\pi : (S^\pm)^* \rightarrow H$

**Idea of Pf:** Generators for $H$

$\iff$ circuits in $M$

- Finitely many since $M$ is finite.

**Example.**

$H = \langle a^2, b \rangle$

Circuits: $a^2, a, b, b^{-1}, ba^2, b^{-1}a^2, b^{-1}a, ba^{-2}$ & $b, a^2$ generate $H$!
Freely reducing a language

**Lemma 3.** \( L = \) reg. lang over \( S^+ \)

\( R = \) lang obtained from \( L \) by freely reducing.

Then \( R \) is regular.

**Pf.** Say \( L \) given by FSA \( M \).

If we see \( s \xrightarrow{s} s' \xrightarrow{s''} s''' \),

add empty edge

\( \rightarrow M' \)

\( M' \) accepts all the words \( M \) did plus their freely reduced versions.

Let \( K = \) language of all freely reduced words in \( S^+ \)

\( K \) is regular (exercise)

\& \( R = K \cap L(M') \)

By Thm, \( R \) regular \( \square \)
Pf of Howson's thm

**H, K** fin gen. subgps of \( F_n \)

**H, K** are images of reg lang's

**Hk, Hk** by Thm.

By Lemma 3 we may assume

**Hk, Hk** consist of freely red.

words, which are exactly elts

of **H, K** (need a free gp for this!)
We can now convert $\mathcal{M}_\varepsilon$ into a deterministic automaton, $\mathcal{D}$. The states of $\mathcal{D}$ consist of all the subsets of $V(\mathcal{M}_\varepsilon)$. The single start state of $\mathcal{D}$ is the subset of $V(\mathcal{M}_\varepsilon)$ consisting of all the start states of $\mathcal{M}_\varepsilon$. The accept states of $\mathcal{D}$ are the subsets of $V(\mathcal{M}_\varepsilon)$ that contain at least one accept state of $\mathcal{M}_\varepsilon$. In $\mathcal{D}$ there is an edge from $U$ to $U'$ labelled by $x$ if, for each $v \in U$, there is an edge labelled $x$ from $v$ to some $v' \in U'$, and $U'$ is entirely composed of such vertices. That is, there is an edge labelled $x$ from $U$ to the vertex corresponding to the set

$$U' = \{v' \in V(\mathcal{M}_\varepsilon) \mid v' \text{ is at the end of an edge labelled } x \text{ that begins at some } v \in U\}.$$
Announcements Mar 30

- Cameras on
- HW due Thu 3:30
- Office Hours Fri 2-3 apt
- Outline Apr 2 ~1 page, teams
- First draft Apr 9
- Makeup points

Today
- Lamplighter groups
- Diestel - Leader graphs
Lamplighter group \((\mathbb{OHGT})\)

Infinite street \(\mathbb{Z}^0\)

\[ L = \mathbb{Z} \times \left( \bigoplus_{0}^{\infty} \mathbb{Z}/2 \right) \text{ as a set.} \]

\[ = \{ (k, \tilde{x}) : k \in \mathbb{Z}, \tilde{x} \in \bigoplus_{0}^{\infty} \mathbb{Z}/2 \} \]

\[ = \{ \text{configurations} \} \text{ or} \]

\[ = \{ \text{actions} \} \]

Multiplication: stack and add

\[ + \]

\[ \rightarrow \]

\[ \rightarrow \]

\[ \text{II} \]
Generators

$t$:  

$a$:  

$t^3 a t^{-3}$

Presentation

$L = \langle a, t \mid a^2 = \text{id}, (t^i a t^{-j})(t^k a t^{-k}) = (t^k a t^{-k})(t^i a t^{-j}) \rangle$
A faithful representation $\varphi$

First a notation for $\bigoplus_{n=0}^{\infty} \mathbb{Z}/2$. 

$\mathbb{Z}/2[t,t^{-1}]: \{\text{polys in } t, t^{-1}\}$

$t^{-2} + 1 + t^5 \in \mathbb{Z}/2[t,t^{-1}]$

$\longleftrightarrow (0,1,0,1,0,0,0,0,1,0,\ldots)$

$0^{th}$ entry

$L = \{(k, \bar{x})\}$

$= \{(k,p) : k \in \mathbb{Z}, p \in \mathbb{Z}/2[t,t^{-1}]\}$

$\varphi : L \rightarrow \text{GL}_2(\mathbb{Z}/2[t,t^{-1}])$

$(k,p) \mapsto \begin{pmatrix} t^k & p \\ 0 & 1 \end{pmatrix}$

Thm. $\varphi$ is a faithful rep.

Pf. inj: clear...

homom: Check relations

$\alpha^2 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = I$

other reln: shears commute.

$\Box$
Example

\[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}
\]
See OHGCT for a discussion of 1D-

- word length (traveling salesman problemish)
- dead ends
- generalize

1. $L_n$: lamps have $2/n$ states

2. Wreath products

$L = \mathbb{Z}_2 \ast \mathbb{Z}$

$G \ast H$ is the lamplighter group with "map" $H$ (like $\mathbb{Z}$ in $L$)

"lamp states" $G$ (like $\mathbb{Z}/2$ in $L$)

$\mathbb{Z}/2 \ast \mathbb{Z}^2$ 2D-traveling salesman problem. HARD!
A little more:

\[ G \wr H = \{(k, \bar{x}): k \in H, \bar{x}: H \to G\} \]

\[ H \times (\bigoplus_H G) \]

1st factor permutes coords of 2nd.

Example

\[ \mathbb{Z}/2 \wr \mathbb{Z} = \{(k, \bar{x}): k \in \mathbb{Z}, \bar{x}: \mathbb{Z} \to \mathbb{Z}/2\} \]

\[ \mathbb{Z} \times (\bigoplus_{\mathbb{Z}} \mathbb{Z}/2) \]

First factor permutes coords of 2nd.
Next: Cayley graph for $L$

Diestel-Leader Graphs

An old question: is every graph quasi-isometric to a Cayley graph?

$DL(m,n)$ is a graph

D-L conjectured

$m \neq n \Rightarrow DL(m,n)$ is not QI to a Cayley graph.
(proved by Eskin-Fisher-Whyte)

But $DL(n) = DL(n,n)$ is the Cayley graph for $L_n$ (lamp gp w/ $n$ states)
DL(2): vertices \( \{(v,w) : v \in V(T_1), w \in V(T_2), h(v) + h(w) = 0\} \)

edges \( (v,w) \) when \( v \rightarrow v' \) in \( T_1 \)
\( w \rightarrow w' \) in \( T_2 \)
**Thm.** $DL(2)$ = Cayley gr. for $L$ wrt $t$, at

How to get a $(k, \vec{x})$ from a vertex of $DL(2)$?

$k$ is easier: $h(v)$

Lamp states on streets $< K$

Make up path from $v$, down path from $w$, concat $\rightarrow \infty$ string of $L/R$. 

Lamp states on streets $= K$
Announcements Apr 1

- Cameras on
- HW due Thu 3:30
- OH Fri 2-3, Tue 11-12, appt
- Outline due Fri nite
- Makeup points

Today
Thompson's group.
Thompson's gp $F$

$F =$ group of assoc. laws

or: How to get between all parenthesizations of an expression

$x_0 : a(bc) \rightarrow (ab)c$

$x_1 : a(b(cd)) \rightarrow a((bc)d) \quad \text{example: } a\left(b\left(c(\text{de})\right)\right) \rightarrow (ab)(c(\text{de}))$

$x_2 : a(b(c(de))) \rightarrow a(b(cd)e))$

Composition: do first, then the second.

Must allow for interpreting $a, b, c$ as expressions themselves and allow expansions

$\cdots a \cdots \rightarrow \cdots (a,a_2) \cdots$

$A, B, C \rightarrow (A)(BC) = (AB)(C)$

$A, B, C$ expressions in $a, b, c$. Subscript $\leftrightarrow \text{“depth”}$
A relation:

\[ a(b(c(d(e)))) \xrightarrow{x_0} (ab)(c(d(e))) \]

\[ x_2 \downarrow \quad \downarrow x_1 \]

\[ a(b((c)d)e)) \xrightarrow{x_0} (ab)((c)d)e) \]

So: \[ x_1x_0 = x_0x_2 \quad \text{(right to left)} \]

More generally:

\[ x_n x_i = x_i x_{n+1} \quad i < n \]
$F$ via PL homeos

$F = \{ \text{orientation preserving, piecewise linear homeomorphisms of } [0,1] \text{ with dyadic break points with slopes powers of } 2^k \}$

under fn composition.

Homeos

A fn $f: [0,1] \to [0,1]$ is a homeo if it is contin with contin. inverse.

$f, g$ are both homeos.

Orient. pres A homeo $f: [0,1] \to [0,1]$ is or. pres. if $f(0) = 0$.

Piecewise linear What you think.

(finitely many line segments)
$F$ via PL homeos

$F = \{ \text{orientation preserving, piecewise linear homeomorphisms of } [0,1] \text{ with dyadic break points with slopes powers of 2} \}$

under fn composition.

---

Breakpoints

Dyadic $\frac{m}{2^k}$ $k, m \in \mathbb{Z}$

Why is this a group?

think about composition or pres.

homeo ✓

break pts dyadic: exercise.

slopes powers of 2: chain rule.
Examples

What is $x, x_0$?

Where is $15/16$?

Break pts: Break pts of $x_0$ \[ \bigcup x_0^{-1}(\text{Break pts of } x_1) \]

\[
\frac{1}{2}, \frac{3}{4} \bigcup x_0^{-1}(\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}\})
\]

\[
\{\frac{3}{4}, \frac{7}{8}, \frac{15}{16}\}
\]
A tree pair is a pair of binary trees with same # of leaves

\[ (S_2, T_2) \cdot (S_1, T_1) \]

1. If \( T_2 = S_1 \), then \( (S_2, T_1) \)
2. If \( T_2 \neq S_1 \), add carets until they are equal.

Reduced if no canceling carets:
Multiplication:

\[(S_2, T_2) \cdot (S_1, T_1)\]

Is \((S_2, T_1)\) if \(T_2 = S_1\)

If \(T_2 \neq S_1\), add carets until they are equal.

Carets ↔ parentheses.

\[\frac{a \cdot (b \cdot c)}{T} \leftrightarrow \frac{(ab) \cdot c}{T}\]

Dividing interval in half
Some Facts about $F$

1. $F$ is generated by $x_0, x_1$.
2. $F$ is finitely presented.
3. $F$ contains $x_0^2 x_1^2$.

Major open question

Q. Is $F$ amenable?

A group is amenable if its Cayley graph admits a Ponzi scheme.

Yes $\Rightarrow$ $F$ is a fin pres amen gp that is not elem. amenable.

No $\Rightarrow$ $F$ is a fin pres non-amen gp with no free subgp.
Announcements Apr 6

- Cameras on
- HW due Thu (your choice of 2)
- Draft due Fri
- Office Hours moved Wed 2-3, Tue 11-12, appt
- Makeup work

Today
Quasi-isometries
**GEOMETRY vs. ALGEBRA**

**Thm.** \( G \cong \mathbb{Z} \Rightarrow \) \( G \) has finite index subgp \( H \cong \mathbb{Z} \).

geometry
algebra
Two Cayley graphs for $\mathbb{Z}$

$S = \{1, 7\}$

$S = \{2, 3\}$

$d(-2, 5) = 7$

$d(-2, 5) = 3$

Looks like $\mathbb{R}$ from far away.

Will show: it is QI to $\mathbb{R}$. 
**Metric space**

\((X, d_X) \quad (Y, d_Y)\) metric spaces.

meaning \(d_X(x_1, x_2)\) is **distance**.

examples: graphs

\[
d(a,c) = 2 \quad d(a,e) = 3 \frac{1}{2}
\]

groups word metric/distance in Cayley graph.
Isometries

Equivalence of metric spaces

If $f: X \rightarrow Y$ is an isometric embedding if it doesn't change distances:

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

If $f$ is also surjective, say $f$ is an isometry.

Examples:

- Two: $f: \cdots \rightarrow \# \cdots$

- Not an isom. emb.: $\cdots$  

- Not: $\text{far in } \mathbb{Z}^2 \rightarrow \text{close in } \mathbb{Z}^2$
Examples

1. \( f: \ldots \rightarrow \ldots \)

Two

\[ \times \]

An isom. emb.

Not:

for \( i \in \mathbb{Z} \)

close in \( \mathbb{Z} \)

or any non-inj.

2. \( \mathbb{Z}^2 \rightarrow \mathbb{R}^2 \)

Inclusion

Is this an isom. emb?

No with std metric on \( \mathbb{R}^2 \)

Yes with "taxicab metric" on \( \mathbb{R}^2 \)

3. Identity \( \mathbb{Z} \rightarrow \mathbb{Z} \)

where first \( \mathbb{Z} \) has \( \{1, 3\} \) metric

second \( \mathbb{Z} \) has \( \{2, 3\} \) metric

Not an isom. emb.
**Bi-Lipschitz equivalence**

$f: X \to Y$ is a bi-Lipschitz embedding if \( \exists K \geq 1 \) s.t.

\[
\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)
\]

(K independent of \( x_1, x_2 \)).

If \( f \) also surj. then \( f \) is bi-Lip equiv.

**Thm.** \( G = \text{group} \)

\( S, S' \) two finite gen sets

\( \text{id: } (G, ds) \to (G, ds') \)

is a bi-Lip eq

**Example**

\( \text{equiv. rel.} \)

\( (\mathbb{Z}^2, \text{std}) \to (\mathbb{R}^2, \text{std}) \)

\( K = \sqrt{2} \) bi-Lip emb
Thm. Let $G$ be a group.
$S, S'$ be two finite generating sets.
$id: (G, ds) \rightarrow (G, ds')$ is a bi-Lipschitz equivalence.

Proof idea: What is $K$?

$$K = \max \left\{ ds'(id, s) : s \in S \right\}$$

$$\bigcup \left\{ ds(id, s) : s \in S' \right\}$$

Use triangle inequality.
Quasi-isometries

\[ f : X \to Y \text{ is a q.i. if } \exists K \geq 1, C \geq 0 \text{ s.t.} \]

\[
\frac{1}{K}d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq Kd(x_1, x_2) + C
\]

If \( \exists D \geq 0 \text{ s.t. each pt of } Y \text{ is within } D \text{ of } f(X) \)
then \( f \) is a quasi-isometry.

Examples. ① \((\mathbb{Z}^2, \text{std}) \to (\mathbb{R}^2, \text{std})\)
\[ K = \sqrt{2}, \quad C = 0, \quad D = \mathbb{R}^2/2 \]

② \((\mathbb{R}^2, \text{std}) \to (\mathbb{Z}^2, \text{std})\)
\[ K = \sqrt{2}, \quad D = 0. \]

③ \((x, y) \mapsto (Lx, Ly)\)
\[ C = \sqrt{2}. \]
$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \begin{cases} 
5x & x \in \mathbb{Q} \\
3x + 1 & x \notin \mathbb{Q}
\end{cases}$

$K = 5 \quad C = 1 \quad D = 1.$

(or 0)

$f(x) = \begin{cases} 
5x & x \neq 0 \\
7 & x = 0
\end{cases}$

$K = 5 \quad C = 7 \quad D = 1$
Ponzi

Yes. Ponzi scheme.

$QI$'s are violent, but still have them: $G \cong \mathbb{Z}$ then $G$ has $\frac{H \cap G}{\mathbb{Z}}$
ANNOUNCEMENTS APR 8

- Cameras on
- HW due Thu - I forgot again!
- First draft due Fri - share on Teams & Reviews
- Office Hours Tue 11, appt.
- Makeup points

Today
- Milnor-Schwartz Lemma
- $G \cong \mathbb{Z} \Rightarrow G$ is virtually $\mathbb{Z}$
Quasi-isometries

\((X, d_X), (Y, d_Y)\) metric spaces

\(f: X \to Y\) is a quasi-isom. if

\[
\frac{1}{k} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq k d_X(x_1, x_2) + C
\]

and there is a \(D\) so all pts of \(Y\) are within distance \(D\) of \(f(X)\)

\(F = \text{finite gp}\)

\(\cong \\ \ \ \Gamma = \bullet\) with \(k = 1\)

\(C = \text{diam}(F)\)

\(D = 0\)
Milnor-Schwarz Lemma (Fund Lemma of GGT)

Thm. \( G \bowtie \Gamma = \text{graph} \)

finite fund dom
(or \( \Gamma / G \) finite)

Action is prop. disc.

Then: \( G \) is fin. gen.

\& \( G \cong \Gamma \).

Graph version

\( G \bowtie \Gamma \) is properly discontinuous
if \( \forall K \subseteq \Gamma \) finite subgraph

\[ \# \{ g \in G : g \cdot K \cap K \neq \emptyset \} < \infty. \]

P.D. and finite f.d. Needed for Thm because...

might \( G \bowtie \Gamma \) trivially.

Example. \( \odot \) \( F=\text{finite gp} \) \( \Gamma = \cdot \implies F \cong \odot \)

with \( K = \)

So: All finite gps are \( \cong \) to trivial gp.
Milnor-Schwarz Lemma
(Fund Lemma of GGT)

Thm. \( G \subset \Gamma = \text{graph finite fund dom} \)
(or \( \Gamma / G \) finite)
Action is prop. disc.

Then: \( G \) is fin. gen.
& \( G \cong \Gamma \).

Examples. 1) \( \mathbb{Z} \Gamma \)
2) \( SL_2(\mathbb{Z}) \subset \text{Farey tree} \).

Prop. disc: a vertex of \( \Gamma \)
is a basis for \( \mathbb{Z}^2 \)

\( g \Gamma \) means \( g \) took a basis in \( \mathbb{K} \) to another
basis in \( \mathbb{K} \).

\( \Rightarrow SL_2 \mathbb{Z} \cong \mathbb{T}_3 \cong \mathbb{T}_4 \cong \mathbb{F}_2 \cong \mathbb{F}_k \) \( k \geq 2 \)
Applications

1. $H \leq G$, finite index.
   $H \cup \Gamma = \text{Cayley graph for } G$.
   with finite fund. dom
   & prop. disc (b/c $G$ acts p.d.)

   $\Rightarrow H \sim_{\alpha} G$.

2. $N \leq G$, $N$ finite.
   $G C_G \Gamma = \text{Cayley graph for } G/N$.
   with p.d., & finite fund dom.

   $\Rightarrow G/N \sim_{\alpha} G$ b/c $N$ is finite.

3. $(G, S) \sim_{\alpha} (G, S')$
Gromov's Program

Which fin. gen. gps are quasi-isometric?

We saw: $G, H$ differ by finite groups $\Rightarrow G \cong H$

We say $G$ is quasi-isometrically rigid if $G \cong H \Rightarrow G, H$ differ by finite gps.

Examples

① Trivial gp.
② $\mathbb{Z}^n$ (we'll prove $n=1$ case soon)
③ Braid groups.
④ Mapping class groups.
⑤ Free gps.
Idea of Milnor-Schwarz

Choose vertex \( v \).

\( R > 0 \) so \( B_R(v) \) contains fundamental domain.

Get the finite gen set as usual

\[ S = \{ g : g \cdot B_R \neq \emptyset \} \]

Finite by prop disc.

What are \( K, C, D \)?

Distances in \( \Gamma \) not much longer than in \( G \) by defn of \( R \).

Opposite direction: If have \( g_i \) with \( |g_i| \to \infty \) & \( g_i \cdot v \) close to \( v \), violate PD.
**Thm.** $G = \text{fin. gen. gp.}$

$G \cong \mathbb{Z}$

$\Rightarrow \exists H \leq G, H \cong \mathbb{Z}$

**Pr.** Let $f: G \to \mathbb{Z}$.

$$\frac{1}{Kd(x,y)} - C \leq |f(x) - f(y)| \leq Kd(x,y) + C$$

WLOG $f(id) = 0$.

(change $C$ if needed).

First Goal: 2 fingers, not 5

one left one right.
Step 1. $G$ has $\infty$ order elt $a$.

Step 2. $|G/\langle a \rangle| < \infty$

For step 1, find $A \subseteq G$, $a \in G$ s.t. $a \cdot A \subseteq A$

$\Rightarrow |a| = \infty$.

How to find $A$?

Let $L \gg K, C$ \quad ($L = k + C$)

$B = f^{-1}([-\frac{L}{2}, \frac{L}{2}])$

The diagram shows a group $G$ with elements labeled $B$, $\text{id}$, $A$, and $f$. The group elements are connected by arrows and the set $[-\frac{L}{2}, \frac{L}{2}]$ is mapped into the group $G$.
Announcements Apr 13

- Cameras on
- Last HW due Thu
- Peer evaluations due Fri
- Presentations next week ~20
- Final draft due Apr 27 3:30.

- Makeup problems
- CIOS

Today
- \( G \tilde{\times} \mathbb{Z} \Rightarrow G \tilde{\times} \mathbb{Z} \)
- Ends of groups:
  - Freudenthal–Hopf Thm
**Thm.** $G = \text{fin gen. gp}$

$G \cong \mathbb{Z}$

Then $G$ has finite index subgp $H \cong \mathbb{Z}$.

**Pf.** Let $f: G \to \mathbb{Z}$ q.i.

$\frac{1}{k}d(x,y) - C \leq |f(x) - f(y)| \leq kd(x,y) + C$

Also: D...

WLOG $f(id) = 0$.

---

**Step 1.** $G$ has $\infty$ order elt a

**Step 2.** $\langle a \rangle$ has finite index in $G$. 
WLOG \( G \setminus B \) has only unbounded pieces (if not, add any bounded pieces to \( B \))

Let \( A_+ = f^{-1}(L_2, \infty) \setminus B \)

Let \( A_- = f^{-1}(-\infty, -L_2) \setminus B \)

Want this pic:

\[
\begin{align*}
\text{A-} & \quad \text{B} \quad \text{A+} \\
\end{align*}
\]

or: \( A_+ \) and \( A_- \) connected, each also separate from one another.

Step 1. \( G \) has \( \infty \) order elt \( a \)

Suffices to find \( A \subseteq G, \ a \in G \)

\[ a \cdot A \nsubseteq A \quad \text{(ping pong)} \]

Let \( L = K + C \) (\( e \) = edge in \( G \))

\[ \Rightarrow f(e) \text{ has length } \leq L \]

\[ B = f^{-1}([-L_2, L_2]) \]

Note: if \( g, h \) connected in \( G \), can’t lie on opp. sides of \( B \).
Let \( L = K + C \) (\( e \) = edge in \( G \) \( \Rightarrow f(e) \) has length \( \leq L \))

\[ B = f^{-1}([-L/2, L/2]) \]

Note: if \( g, h \) connected in \( G \), can't lie on opp. sides of \( B \).

Claim 1. \( G \setminus B \) has \( \geq 2 \) pieces
\( \Rightarrow \) \( A_+ \), \( A_- \) not connected to each other.

Pf. The above note.

Claim 2. \( G \setminus B \) has \( \leq 2 \) pieces.

Pf. Otherwise find arbitrarily far pt of \( G \) mapping to same pt of \( \mathbb{Z} \).

Now Have:

\[ A_+ = f^{-1}(L/2, \infty) \setminus B \]
\[ A_- = f^{-1}(-\infty, -L/2) \setminus B \]
Let $g, h \in G$ s.t. $g \in A^+, g_- \in A^-$.

$d(id, g) > 2 \text{diam}(B)$

Claim 2. \( \exists \tilde{D} \text{ s.t. } D \text{ nbd of } \langle a \rangle \text{ in } G \) is $G$.

Claim 3. \(|G/\langle a \rangle| < \infty\).

Pf of Claim 1: By Step 1, $a^n$ all distinct, but $G$ locally finite (where we use $G$ fin gen.).

Pf of Claim 2. Claim 1 \( \Rightarrow \)

\[
d(la^m, a^n) \to \infty \quad \text{as } \mid m-n \mid \to \infty
\]

\[
\Rightarrow f(a^m) \to \infty \quad f(a^{-n}) \to -\infty
\]
Claim 2. Let $D$ be the $D$-neighborhood of $\langle a \rangle$ in $G$. Then $G$ is $G$.

Proof of Claim 2. By Claim 1, $d(a^m, a^n) \rightarrow \infty$ as $|m-n| \rightarrow \infty$.

$\Rightarrow f(a^i) \rightarrow \infty$, $f(a^{-i}) \rightarrow -\infty$

If there were points in $G$ arbitrarily far from $\langle a \rangle$ then arbitrary far from $\langle a \rangle$, then arb for points in $G$ would map to "same" pt in $\mathbb{Z}$.

Claim 3. $|G/\langle a \rangle| < \infty$.

Let $\Gamma$ be the Cayley graph for $G$ such that $\Gamma/\langle a \rangle$ has one vertex for each $\langle a \rangle$ and locally finite, and finite diameter by Claim 2.

$\Rightarrow \Gamma/\langle a \rangle$ finite.

But vertices of $\Gamma/\langle a \rangle$ are the cosets of $\langle a \rangle$ in $G$. $\square$
Ends of Groups

Freudenthal-Hopf Thm

\[ G = \text{fin gen gp} \]

\[ \Rightarrow G \text{ has 0, 1, 2, or (\infty many) ends} \]

Some defs:

\[ \Gamma = \text{connected graph, locally finite.} \]

\[ V = \text{base vertex.} \]

\[ B_n = \text{ball of radius n around } V. \]

\[ \| \Gamma \backslash \text{Ball} \| = \# \text{ unbounded pieces of } \Gamma \backslash \text{Ball} \].

\[ e(\Gamma) = \lim_{n \to \infty} \| \Gamma \backslash B_n \| \]
\[ \| \Gamma / B_n \| = \# \text{ unbounded pieces of } \Gamma / B_n. \]

\[ e(\Gamma) = \lim_{n \to \infty} \| \Gamma / B_n \|. \]

**Examples**

1. \( \Gamma \) finite.
   \[ \Rightarrow e(\Gamma) = 0 \]

2. \( \Gamma = \)
   \[ \Rightarrow e(\Gamma) = 2 \]

3. \( \Gamma = \)
   \[ \Rightarrow e(\Gamma) = 1 \]

4. \( \Gamma = \)
   \[ \Rightarrow e(\Gamma) = \infty \]

5. \( \Gamma = \)
   \[ \Rightarrow e(\Gamma) = n. \]
Announcements Apr 15

- Cameras on
- Peer evaluations due Fri/Sun
- Presentations next week ~20 mins
- Final draft due Apr 27 3:30
- Office Hours Fri-postponed.
- Makeup problems
- CIOS

Today
- Ends of groups: Freudenthal-Hopf Thm
- Summary
Ends of Groups

Freudenthal-Hopf Thm

\[ G = \text{fin gen gp} \]
\[ \Rightarrow G \text{ has } 0, 1, 2, \text{ or } (\infty \text{ many}) \text{ ends} \]

Some defs:

\[ \Gamma \] = connected graph, locally finite.
\[ V \] = base vertex.
\[ B_n \] = ball of radius \( n \) around \( V \).

\[ \| \Gamma \setminus B_n \| = \# \text{ unbounded pieces of } \Gamma \setminus B_n. \]

\[ e(\Gamma) = \lim_{n \to \infty} \| \Gamma \setminus B_n \|. \]
\[ \| \Gamma \setminus B_n \| = \# \text{ unbounded pieces of } \Gamma \setminus B_n. \]

\[ \| \Gamma \setminus B_n \| = 5 \]

\[ e(\Gamma) = \lim_{n \to \infty} \| \Gamma \setminus B_n \|. \]

**Examples**

1. \( \Gamma \) finite.
   \[ \Rightarrow e(\Gamma) = 0 \]

2. \( \Gamma = \]
   \[ e(\Gamma) = 2 \]

3. \( \Gamma = \]
   \[ e(\Gamma) = 3 \]

4. \( \Gamma = \]
   \[ e(\Gamma) = \infty \]

5. \( \Gamma = \]
   \[ e(\Gamma) = n. \]
\( e(\Gamma) \) is well defined

**Lemma.** \( \| \Gamma \setminus B_n \| \) is a non-decreasing seq. of pos. integers.

**Pf.**

When you take a unbdd subgraph and remove a banded subgraph \((B_{n+1} \setminus B_n)\), it becomes \( \geq 1 \) unbounded piece (also, pieces can't merge when you remove stuff).

**Cor.** \( e(\Gamma) \) is well-def.

Next goal: \( e(\Gamma) \) is a QT invariant.
Alternate defn:

\[ e_c(\Gamma) = \sup \{ \|\Gamma \setminus C\| : C \subseteq \Gamma \text{ finite} \} \]

Lemma. \( e_c(\Gamma) = e(\Gamma) \).

Pf.

\[ \geq \] sup over bigger set.

\[ \leq \] Any such \( C \) is contained in a \( B_n \). Use argument from last slide.

\[ C \subseteq B_n, \quad \|\Gamma \setminus B_n\| \geq \|\Gamma \setminus C\|. \]
# Ends is a QI invariant

Prop. If $\Gamma_1 \eqq_{QI} \Gamma_2$ then $e(\Gamma_1) = e(\Gamma_2)$.

**Proof.** Let's convince ourselves that $e(\Gamma_1) = 1 \iff e(\Gamma_2) = 1$.

Assume $e(\Gamma_1) = 1$.

Assume $e(\Gamma_1) = 1$.

Assume $B_n$ cuts $\Gamma_2$ in two unbounded pieces.

Thus they are connected by path $P$ outside $B_n$.

Choose big $N$.

Contradiction.

Want: $x, y$ connected outside $B_n$. Note: $f(x), f(y)$ outside $B_n$. 

Assume $B_n$ cuts $\Gamma_2$ in two unbounded pieces.
Poll

Q. How many ends does braid gp $B_n$ have?

$B_1 \cong 1 \Rightarrow e(B_1) = 0$

$B_2 \cong \mathbb{Z} \Rightarrow e(B_2) = 2$

$B_3 \text{ ???}

A. $e(B_n) = 1 \quad n \geq 3$.

Step 1.

Pf. $e(B_n) = e(PB_n)$

since $[B_n:PB_n] = n! < \infty$

Step 2. $PB_n \cong PB_n/\mathbb{Z} \times \mathbb{Z}$

Fact. If $G, H$ infinite, then $e(G \times H) = 1$. 

#
Freudenthal-Hopf Thm

$G = \text{fin gen.}$

Then $e(G) \in \{0, 1, 2, \infty\}$.

Assume 3 ends

$\Rightarrow 4$ ends etc.,
Some Ends we know

\[ e(\mathbb{Z}) = 2 \]
\[ e(\text{finite gp}) = 0 \]
\[ e(\mathbb{Z}^n) = 1 \quad n \geq 2 \]
\[ e(F_k) = \infty \quad k \geq 2 \]
\[ e(\mathbb{B}_n) = 1 \quad n \geq 3 \]
\[ e(\text{SL}_2 \mathbb{Z}) = \infty \]

\[ e(W_{333}) = 1 \]

Different # ends
\[ \Rightarrow \text{not quasi-isometric.} \]
Also sometimes gps with same # of ends are not QT.
example \( \mathbb{Z}^n, \mathbb{Z}^n \) m \neq n.
(different growth rates)
Who cares if groups are not QI?

Milnor-Schwarz: If $G$ acts geometrically on $\Gamma$
then $G \cong \Gamma$.

So, if $G, H \cong \Gamma$ then $G \cong H$.

Infinitely many: $B_n$ does not act geometrically on $\mathbb{R}$, or tree.

Corollary: $SL_2 \mathbb{Z}$ does not act geometrically on $\mathbb{R}^2$. 
Geometry, Topology, and Group Theory

Last time: We’ve learned so much!

Certain groups can/cannot act (geometrically) on the same graph/space.

Today: There’s so much more to learn!
Hyperbolic Geometry

Euclid's Postulates ①-④ boring.

⑤ Given a point $P$ not on line $L$, there exists a line $L'$ through $P$ and not intersect $L$.

Lobachevsky/Poincaré: There is geometry without ⑤ → Hyperbolic plane
Hyperbolic Plane $\mathbb{H}^2$

Defn 1
Compare Farey Graph.

Defn 2

The straight lines are pieces of circles/lines perpendicular to the boundary. => metric is a multiple of one below

Riemannian geometry

Metric:
Euclidean metric
$(1-r^2)$

=> straight lines almost same as Eucl. dist.
all triangles congruent in $\mathbb{H}^2$
all have interior angles $0$
(all triangles “skinny”)

Farey graph

sum of interior angles $< \pi$

Compare spherical geometry

interior angles $> \pi$
Which groups act on $1H^2$?

For $E^2$ have reflection groups, e.g. $W_{333}$ and $\mathbb{Z}^2$ all of these coming from tilings

Let's look for tilings of $1H^2$. 
Looking for tiles in $\mathbb{H}^2$

small $n$-gons have nearly Euclidean interior angles
sums $\pi(n-2)$

$17T \Rightarrow$ regular right angled pentagon!

Now tile!
Aside: Defn #3 of $H^2$.

Isometries are:

$$\{ \text{M"obius transformation preserving open unit disk} \}$$

$$\left\{ f(z) = \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R}, \right\}$$
reflection group
\[ \langle x_1, \ldots, x_5 : (x_1 x_2)^2 = (x_2 x_3)^2 = \cdots = (x_5 x_1)^2 = \text{id} \rangle \]

Now have many new gps, not QT to Euclidean gps W333 etc.
Connection to Topology

$H^2$

\[\begin{array}{c c c c c c}
    & a & b & a & b & a \\
\hline
b & a & b & a & b & \\
a & b & a & b & \\
b & a & b & \\
\end{array}\]

\[\text{glue } b \rightarrow \text{glue } a\]

\[\begin{array}{c c c c c c}
    a & b & \rightarrow & a \\
\end{array}\]

\[\text{the loop around the square is the relation}\]

\[\langle a, b : aba^{-1}b' = \text{id} \rangle \]

\[\cong \mathbb{Z}^2 \quad \text{QI to } \mathbb{H}^2\]

$H^2$

right angled octagon

\[\langle a, b, c, d : \text{fundamental gp of } S_2 \rangle \]

$S_2$

Surface of genus 2.

\[\text{QI to } H^2\]
Milnor-Schnorrz:
fund. gp of $S_2 \cong \mathbb{H}^2$
Hyperbolic Groups à la Gromov

A space is $\delta$-hyperbolic if for any triangle, the $\delta$-neighborhoods of two sides together contain the 3rd side.

Facts:
- $H^2$ is $\delta$-hyperbolic ($\delta = \log 2$ ?)
- $\delta$-hyp. is a QI invt $\Rightarrow$ fund gp of $S_2$ is $\delta$-hyp.
Two Theorems of Gromov

Thm. Most groups are hyperbolic.

Thm. A group is $\delta$-hyp $\rightarrow$ its word problem is solvable in linear time.
Why does fund gp of $S_2$ have linear time soln to WP?

\[ \langle a, b, c, d : \text{aba}^{-1}\text{b}^{-1}\text{c}^{-1}\text{d}^{-1} \rangle \]

Any closed loop in Cayley graph must use $\geq 6$ sides of a single octagon.

So can replace word of length 6 with word of length 2 SHORTENING.
Here there be dragons.

Key: Ab — abelian, Nilp — nilpotent, PC — polycyclic, Solv — solvable, EA — elementary amenable, F = free, EF — elementarily free, L = limit, Hyp — hyperbolic, $C_0$ — CAT(0), SH — semi-hyperbolic, Aut — automatic, IP(2) — quadratic isoperimetric inequality, Comb — combable, Asynch — asynchronously combable, vNT — the von Neumann–Tits line. The question marks indicate regions for which it is unknown whether any groups are present.