Cup, Cap, and Poincaré Duality

Poincaré duality. \( H^k(X) \overset{\sim}{\to} H_{n-k}(X) \)
\[ \varphi \mapsto [M] \cap \varphi \]

Also. Under this isomorphism, cup product corresponds to intersection: \( \varphi \cup \psi \mapsto \varphi^* \cap \psi^* \)

We'll work with \( \Delta \)-complexes, simplicial (co)homology.

Cap

Idea. Realize cohomology class \( \varphi \) as “intersect with dual object.” Push dual in each simplex toward highest vertex (this is well-defined across different simplices in a \( \Delta \)-complex). Result is \( [M] \cap \varphi = \varphi^* \)

Example 1. \( n = 2, k = 1 \)

\( \varphi \)-values → 2

This is exactly what \( \varphi \cap [M] \) gives!

Note: in a manifold, orientation is same as co-orientation.
Example 2. \( n=2, k=1 \)

Example 3. \( n=3, k=1 \)

Example 4. \( n=3, k=1 \)
**Example 5**  \( n=3, k=2 \)

\[ \begin{array}{c}
\begin{array}{c}
3 \\
-1 \\
2 \\
1 \\
0
\end{array}
\end{array} \quad \text{push} \quad \begin{array}{c}
\begin{array}{c}
2 \\
-1 \\
2 \\
1 \\
0
\end{array}
\end{array} \]

**Cup**

Idea. To find \( \varphi \cup \psi \), push \( \varphi \) up, push \( \psi \) down and intersect

**Example 1.**  \( n=2, k=1 \)

\[ \begin{array}{c}
\begin{array}{c}
2 \\
\varphi = 2 \\
1 \\
0
\end{array}
\end{array} \quad \text{push} \quad \begin{array}{c}
\begin{array}{c}
2 \\
\psi = 2 \\
1 \\
0
\end{array}
\end{array} \]

Intersection
Can view same example in context of nearby triangles:

We can modify the curves by homotopy, giving cohomologous cochains:
Example 2. \( n=3, k=1,1 \) (mod 2 this time)

Have \( \varphi \psi \psi \in H^2 \) should be dual to a 1-cell. If we push all the way and intersect, get a point (not what we want). If we push almost all the way, we get what we want:

Note: In the earlier examples, pushing almost all the way also works.