Andy Putman — No Boundaries Conference — 27 Oct '17

Covers and simple closed curves

Goal: Study $H_1(\tilde{\Sigma})$ for finite covers $\tilde{\Sigma} \to \Sigma$  orientable surfaces.

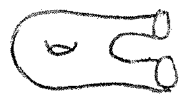
I Motivation

Mod_g^b = mapping class gp of surface of genus g & b bdry components

Q $\exists\ \Gamma < \text{Mod}_g^b$ of finite index such that $H_1(\Gamma;\mathbb{Q}) \neq 0$?

A Yes for $g \leq 2$.
"Degenerate cases – low-genus mcg is very similar to a braid gp"
Open for $g \geq 3$.

Defn For a finite cover $\tilde{\Sigma} \to \Sigma$, let $V_{\tilde{\Sigma}} = \dfrac{H_1(\tilde{\Sigma};\mathbb{Q})}{\langle \text{bdry-components} \rangle}$ ← $\mathbb{Q}$-vector space.

 "some homology is carried by interior and some by bdry – kill off the boundary components by gluing disks to them all"

Defn Let $\text{Mod}_{\tilde{\Sigma}}$ = subgp of $\text{Mod}(\Sigma)$ that lifts to $\tilde{\Sigma}$.

Observe: $\text{Mod}(\Sigma) \geq \text{Mod}_{\tilde{\Sigma}}$ finite index,
$\text{Mod}_{\tilde{\Sigma}} \curvearrowright V_{\tilde{\Sigma}}$.

Thm (P–Wieland) The following are equivalent:

(i) $\forall\ g \geq 3,\ b \geq 0,\ \Gamma < \text{Mod}_g^b$ finite index, we have $H_1(\Gamma;\mathbb{Q}) = 0$. "virtual first Betti number of $\text{Mod}(\Sigma)$ is zero"

(ii) $\forall\ g \geq 3,\ b \geq 0$, finite covers $\tilde{\Sigma} \to \Sigma_g^b$, all nonzero orbits of $\text{Mod}_{\tilde{\Sigma}} \curvearrowright V_{\tilde{\Sigma}}$ are infinite.

Remarks · proof "not (ii)" implies "not (i)" is a direct construction. "A counterexample to (ii) used to cook up a finite index subgp of the mcg that surjects onto $\mathbb{Z}$, and it cooks up a congruence subgp"

"This mysteriously bypasses the congruence subgp property".

The proof of (ii) $\Rightarrow$ (i) uses substantial work of Marco Boggi; ultimately rests on Deligne's theory of weights.

Audience question: You're asking, given a homology class in a finite cover, are there homeomorphisms that lift and move it? Was this studied by Nielsen etc?

Andy: As far as I know the answer is still open.

<u>Remark</u> Work of Farb-Hensel proves Aut(F_n) version of (ii). It is not equivalent to (i). (No analogue of Deligne's theory of weights).

<u>Remark</u> (ii) known for many covers.

- Looijenga – abelian covers.
- Grunewald–Larsen–Lubotzky–Malestein
  – verifies for large class of covers
- still open in general.

they prove that the image of $\mathrm{Mod}_{\tilde{\Sigma}}$ in $GL(V_{\tilde{\Sigma}})$ is a lattice inside some obvious Lie gp in there.

<u>Difficulty</u> It is hard to relate topology of $\tilde{\Sigma} \to \Sigma$ with algebra of $H_1(\tilde{\Sigma})$.

II Simple closed curve (scc) homology

<u>Def^n</u>. For a finite cover $\pi: \tilde{\Sigma} \to \Sigma$, let

$$H_1^{scc}(\tilde{\Sigma}) = \operatorname{span}\left\{ [\tilde{\gamma}] \in H_1(\tilde{\Sigma}) \;\middle|\; \tilde{\gamma} \text{ a component of } \pi^{-1}(\gamma) \text{ for scc } \gamma \subseteq \Sigma \right\}$$

Equivalently, $K < \pi_1(\Sigma)$ finite index, subgp assoc. to cover $\tilde{\Sigma}$.

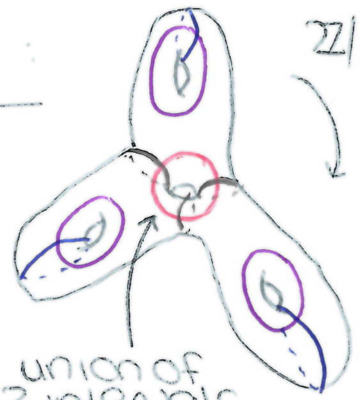
Then $H_1(\tilde{\Sigma}) = H_1(K)$ and

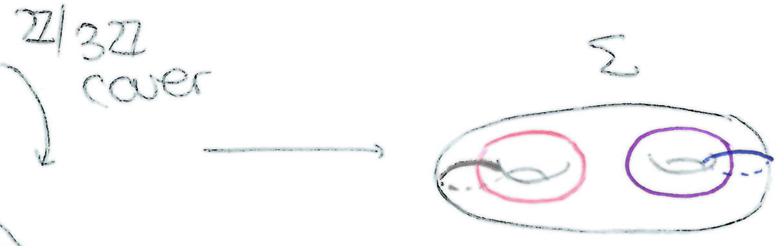
$$H_1^{scc}(\tilde{\Sigma}) = \left\{ [x^k] \in H_1(K) \;\middle|\; \begin{array}{l} x \in \pi_1(\Sigma) \text{ a scc,} \\ \text{and } x^k \in K \end{array} \right\}$$

[Equivalence – an exercise!]

Example:

$\tilde{\Sigma}$



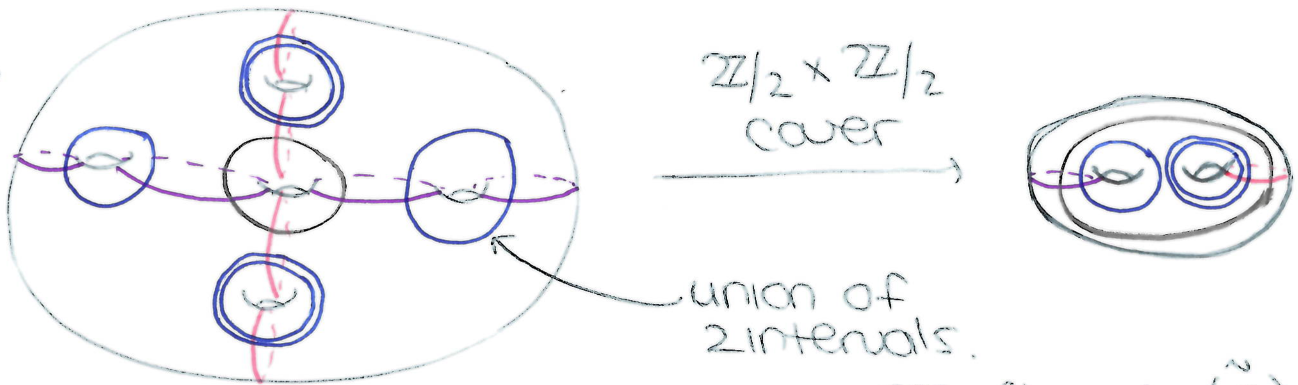


$H_1^{scc}(\tilde{\Sigma}) = H_1(\tilde{\Sigma})$

"we've hit every curve we need".

Example

$\tilde{\Sigma}$



again $H_1^{scc}(\tilde{\Sigma}) = H_1(\tilde{\Sigma})$

Q (Marché, Farb-Hensel, Looijenga) Is $H_1^{scc}(\tilde{\Sigma}) = H_1(\tilde{\Sigma})$ for all finite covers $\tilde{\Sigma} \to \Sigma$?

A No.

Thm (Koberda-Santharoubane) For $\pi_1(\Sigma)$ nonabelian, $\exists$ finite cover $\tilde{\Sigma} \to \Sigma$ with $H_1^{scc}(\tilde{\Sigma}) \neq H_1(\tilde{\Sigma})$.

Pf uses TQFT.

Problem. KS cannot rule out that $H_1^{scc}(\tilde{\Sigma})$ is finite index in $H_1(\tilde{\Sigma})$ (so $H_1^{scc}(\tilde{\Sigma}) = H_1(\tilde{\Sigma})$ w/ $\mathbb{Q}$ coeffs).

Thm (Malestein-P) For $\pi_1(\Sigma)$ free nonabelian, $\exists$ finite cover $\tilde{\Sigma} \to \Sigma$ with $H_1^{scc}(\tilde{\Sigma};\mathbb{Q}) \neq H_1(\tilde{\Sigma};\mathbb{Q})$.

Defn Fix subset $\mathcal{O} \subseteq F_n$. For $K < F_n$ finite index, let $H_1^{\mathcal{O}}(K) = \langle [x^k] \in H_1(K) \mid x \in \mathcal{O},\ x^k \in K \rangle$

Example $\mathcal{O} = \{ \text{scc in } \Sigma \} \subseteq \pi_1(\Sigma)$
then $H_1^{\mathcal{O}}(K) = H_1^{scc}(\tilde{\Sigma})$ for cover $\tilde{\Sigma}$ assoc. to K

<u>Thm</u> (MP) For $n \geq 2$, $\mathcal{O} \subseteq F_n$ contained in finitely many $\mathrm{Aut}(F_n)$-orbits, $\exists$ $K \overset{f.i}{\trianglelefteq} F_n$ with
$$H_1^{\mathcal{O}}(K;\mathbb{Q}) \neq H_1(K;\mathbb{Q}).$$

"This somehow says that there is no hope of writing down generators that are powers of some fixed collection of elements in any sort of uniform way".

III Proof ideas. (for $\mathcal{O} = \{\text{primitive elts in } F_n\}$)

- pf uses a rep-theoretic certificate by Farb-Hensel.
- requires "very strange" $G = F_n / K$.

<u>Prop</u> For $n \geq 2$, p prime, $\exists$ finite p-gp G and a central subgp $C \leq G$, $C \cong \mathbb{Z}/p\mathbb{Z}$ st.
(1) $H_1(G;\mathbb{F}_p) \cong \mathbb{F}_p^n$ "quite large"
(2) $\forall g \in G$ projecting nontrivially to $H_1(G;\mathbb{F}_p)$, have $C \leq \langle g \rangle$.

<u>Example</u> ($n=1$, p arbitrary) $G = C = \mathbb{Z}/p\mathbb{Z}$.

<u>Example</u> ($n=2, p=2$). G = 8-element quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$

$C = \{\pm 1\}$.

All cyclic subgps gen by $\pm i, \pm j, \pm k$ contains C.

<u>Remark</u> In these examples, something stronger holds: $\forall g \in G$ nontrivial, $C \leq \langle g \rangle$. (Atypical behaviour)
This implies G is a <u>Frobenius complement</u>.
(ie, $\exists$ complex G-rep V w no fixed vectors, $V \cong \mathrm{Ind}_C^G \mathbb{C}_c$ where $\mathbb{C}_c = \mathbb{C}$ with C-action $e^{2\pi i/p}$).
$\rightsquigarrow$ fixed-pt-free action on sphere.

These do not exist for $n \geq 3$.

Pf of <u>Prop</u> uses theory of "restricted Lie alg's in char p.