REVIEW OF THURSTON'S WORK ON SURFACES

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In 1976 (incidentally, the year the reviewer was born), Thurston first circulated his now-famous preprint on the classification of surface homeomorphisms [81]. It states that every homeomorphism of a compact surface is homotopic to a homeomorphism in standard form.

The generic standard form is a pseudo-Anosov homeomorphism. Such a homeomorphism acts on the surface by preserving two transverse singular measured foliations, multiplying the measure of one by a stretch factor $\lambda > 1$ and the other by $1/\lambda$. In other words, away from a finite set of singular points, a pseudo-Anosov homeomorphism is modeled on an Anosov map of the torus, or equivalently, an element of $\mathrm{SL}(2, \mathbb{Z})$ with two real eigenvalues.

The precise statement of Thurston's theorem is: every homeomorphism of a compact surface is homotopic to a homeomorphism that either (a) has finite order, (b) is reducible (that is, fixes an essential 1-submanifold), or (c) is pseudo-Anosov. Moreover, the last case is exclusive from the first two. It was realized after Thurston's work (see [63]) that Nielsen had assembled all of the tools needed for the classification [66, 67, 68, 69], and so this theorem is sometimes called the Nielsen–Thurston classification theorem.

Let $\mathrm{Mod}(S)$ be the mapping class group of a surface S, that is, the group of homotopy classes of homeomorphisms of S. Another way to state the classification which suggests an analogy with the Jordan canonical form is: for every element of $\mathrm{Mod}(S)$, there is a representative ϕ and a (possibly empty) ϕ-invariant 1-submanifold C so the restriction of ϕ to $S - C$ has two kinds of components, finite order and pseudo-Anosov. Birman, Lubotzky, and McCarthy showed that the reduction system C is canonical [13].

Shortly after Thurston's announcement, there was a flurry of activity to understand what he did. The book under review is the product of a year-long seminar at Orsay devoted to Thurston's work. The lectures were based on notes from Thurston's graduate course at Princeton University, handwritten by Michael Handel and William Floyd.

In the broadest of strokes, Thurston's proof proceeds as follows. Let S_g be a closed, connected, orientable surface of genus $g \geq 2$. Fricke showed that the Teichmüller space $\mathrm{Teich}(S_g)$, the space of hyperbolic metrics on S_g up to isotopy, is an open ball of dimension $6g - 6$. The key idea of Thurston is that $\mathrm{Teich}(S_g)$ has a compactification that is homeomorphic to a closed ball; the boundary sphere is $\mathrm{PMF}(S_g)$, the space of projective classes of measured foliations on S_g. Moreover, $\mathrm{Mod}(S_g)$ acts on this closed ball. Then, in a most spectacular application of the Brouwer fixed point theorem, Thurston concludes that each element of $\mathrm{Mod}(S_g)$ fixes some point of $\text{Teich}(S_g) \cup \text{PMF}(S_g)$; by analyzing the various cases for the fixed point, we obtain the classification theorem.

The classification theorem is most remarkable for the myriad of proofs it has spawned, each with its own underlying theory and applications. Very quickly upon learning of Thurston's theorem, Bers produced an analytic proof, using Teichmüller's theory of extremal quasiconformal maps [9]. A few years later, Casson gave a proof using hyperbolic geometry, in particular the theory of measured laminations [19]. Handel and Thurston [40] gave an elementary argument for the classification using the action on the boundary of the hyperbolic plane (à la Nielsen). Bonahon then developed the theory from the point of view of geodesic currents; the space of geodesic currents carries a Lorentz-type geometry, with $\text{Teich}(S_g)$ lying on a metric sphere and $\text{PMF}(S_g)$ as the set of rays of the light cone [14]. Finally, in 1992, Bestvina and Handel gave a combinatorial topological approach which used train tracks and yields an algorithm to determine the Thurston type of a homeomorphism [10]; this has been implemented by Brinkmann [16].

The Orsay seminar also consisted of lectures that presented the analytic point of view of Bers. The lectures became the subject of a complementary text by William Abikoff [2]. In the preface to the book under review, the editors state that, in the end, the two points of view were found to be more independent of each other than was initially believed.

1. Some aspects of Thurston's Theory

Besides giving a detailed proof of the classification of surface homeomorphisms, the seminar at Orsay focused on three topics related to the theorem: the construction of pseudo-Anosov maps, their dynamics, and the geometric properties of the associated mapping tori. We review these topics here, making mention of a few more recent advances.

Constructions of pseudo-Anosov maps. Even given the classification theorem, one thing is not at all obvious: do pseudo-Anosov homeomorphisms exist? If so, how do we construct them? Nielsen knew a few examples of infinite order, irreducible mapping classes, but few enough that he conjectured there were none acting trivially on the homology of the surface.

In his original work, Thurston gave a fairly general construction in terms of Dehn twists. Let A and B be two simple closed curves or multicurves (a multicurve is a collection of disjoint simple closed curves) in S_g, and let T_A and T_B denote the mapping classes obtained by performing a positive Dehn twist on each component of A and B. Assuming that A and B are in minimal position and $S_g - (A \cup B)$ is a union of disks, Thurston proved that most elements of the group generated by T_A and T_B are pseudo-Anosov; in particular, $T_A T_B T_A^{-1} T_B^{-1}$ is. Taking A to be a separating curve, we obtain a wealth of counterexamples to Nielsen's conjecture.

Penner gave another Dehn twist construction, again using a pair of multicurves [71]; see also [28, 7]. There is a large overlap between his construction and Thurston's, although neither is subsumed by the other. Penner conjectured that every pseudo-Anosov map has a power that is given by his construction, but there has been very little progress on this question.

Thurston showed that the stretch factor for a pseudo-Anosov homeomorphism of S_g is an algebraic integer whose degree is bounded above by $6g-6$, the dimension of $\mathrm{Teich}(S_g)$. Thurston states that this bound is sharp, but as far as the reviewer knows this is still an open problem. In this direction, Arnoux and Yoccoz [5] were able to use the theory of interval exchange transformations (see [73, 83]) to construct a sequence of pseudo-Anosov maps where the algebraic degrees of the stretch factors go to infinity; specifically, they found examples on S_g with degree g.

It is also possible to construct pseudo-Anosov maps via branched covers. Given any pseudo-Anosov homeomorphism ϕ of S_g, we can choose a finite orbit of arbitrarily large cardinality (see below) and construct a cover $S_h \to S_g$ branched over some subset of the orbit. Some power of ϕ then lifts to a map of S_h that is also pseudo-Anosov. However, this construction is limited, for instance because the stretch factor of the newly constructed homeomorphism is the same as the original.

Casson gave a homological criterion for detecting pseudo-Anosov maps: a mapping class is pseudo-Anosov if the characteristic polynomial $p(t)$ for its action on $H_1(S_g;\mathbb{Z})$ satisfies the following three properties: it is irreducible, it does not have a root of unity as a root, and it is not a polynomial in t^k for $k > 1$ (see [27] for a slight strengthening). Notice that if h satisfies this criterion and f acts trivially on $H_1(S_g)$, then fh is pseudo-Anosov.

In the case of a punctured surface $S_g - p$, there is another construction due to Kra. The Birman exact sequence tells us there is an injective homomorphism $\pi_1(S_g, p) \to \mathrm{Mod}(S_g - p)$ given by pushing p along loops. Kra proved if γ intersects every nontrivial element of $\pi_1(S_g)$ in an essential way, then the image of γ in $\mathrm{Mod}(S_g - p)$ is pseudo-Anosov [50]. His proof uses the complex-analytic point of view of Bers; see [27] for an elementary proof.

A long-standing open question is whether we can construct a pseudo-Anosov homeomorphism whose stretch factor is any prescribed number λ, where λ is an algebraic unit, all of whose conjugates besides $1/\lambda$ have absolute value lying in $(1/\lambda, \lambda)$ [32]. In his final paper, Thurston resolved the analogous question for self-maps of graphs [77].

Recently, Mangahas [57] proved that, given any generating set for $\mathrm{Mod}(S_g)$, there is an explicit pseudo-Anosov element with word length at most K, where K depends only on g; see also [33]. The abundance of pseudo-Anosov mapping classes is underscored by work of Maher [56] and Rivin [74] who showed that, in some sense, the random mapping class is pseudo-Anosov (it is an open question whether pseudo-Anosov elements are generic in the sense that the fraction of pseudo-Anosov mapping classes in the ball of radius R in the Cayley graph for $\mathrm{Mod}(S_g)$ goes to 1 as $R \to \infty$).

Subgroup classification. A finer analysis of the action of the mapping class group on the Thurston compactification of Teichmüller space gives a classification of subgroups of the mapping class group. Ivanov proved a direct generalization of the classification of individual homeomorphisms: every subgroup of $\mathrm{Mod}(S_g)$ is either (a) finite, (b) reducible (that is, there is an essential 1-submanifold whose isotopy class is fixed by each element), or (c) contains a pseudo-Anosov element [46].

McCarthy gave an analog of the Tits alternative for finitely generated linear groups: every subgroup of $\mathrm{Mod}(S_g)$ either contains a free group of rank 2 or contains a finitely generated abelian subgroup of finite index [60]. The key here is the fact that pseudo-Anosov mapping classes act with source-sink dynamics on $\mathrm{PMF}(S_g)$.

Birman–Lubotzky–McCarthy [13] showed that every solvable subgroup of the mapping class group is virtually abelian, and that a free abelian subgroup of $\mathrm{Mod}(S_g)$ has rank at most $3g - 3$. Moreover, they gave a very clear picture of what free abelian subgroups look like: each is (up to finite index) contained in the free abelian group generated by a collection of pseudo-Anosov maps and Dehn twists supported on disjoint subsurfaces.

Handel and Mosher have very recently suggested a clean way to assemble all of these theorems into one convenient package that they call the omnibus subgroup theorem [65]. To state it requires some setup. First, for a given mapping class $f \in \mathrm{Mod}(S_g)$, let N denote a regular neighborhood of its canonical reduction system. The active subsurface A_f is the union of the components of $S_g - N$ on which the first return map for (a representative of) f is pseudo-Anosov, together with the components of N with the property that the components of $S - N$ on either side have periodic first return map. The active subsurface is empty if and only if f is periodic, and is the whole surface if and only if f is pseudo-Anosov.

The omnibus subgroup theorem is: every subgroup of $\mathrm{Mod}(S_g)$ has an element f whose active subsurface A_f is maximal, that is, the active subsurface of every other element is (isotopic to) a subsurface of A_f. One can derive all of the above results about subgroups of $\mathrm{Mod}(S_g)$ from this theorem.

We will mention three open questions about subgroups of $\mathrm{Mod}(S_g)$. Farb asked if every finitely generated normal subgroup is commensurable to either $\mathrm{Mod}(S_g)$ or its Torelli subgroup [24]. Ivanov asked if for large d the group generated by all dth powers of Dehn twists has any relations besides the obvious ones [48]. Ivanov also asked if every finite index subgroup of $\mathrm{Mod}(S_g)$ contains a congruence subgroup, that is, a subgroup arising from a cover $S_h \to S_g$ (not long ago, Thurston resolved the $g = 0$ case [62]).

Recently, Dahmani, Guirardel, and Osin resolved a long-standing open question about subgroups of $\mathrm{Mod}(S_g)$ by showing that there are normal subgroups where every nontrivial element is pseudo-Anosov [21] (see also [84]). Such subgroups even arise as the normal closure of a single element.

Fibered 3-manifolds. Thurston first arrived at the theory of pseudo-Anosov homeomorphisms while trying to determine which 3-manifolds admit hyperbolic structures. In his own words [76]:

> *It seemed pretty obvious that a 3-manifold that fibers over the circle couldn't possibly have a hyperbolic structure. I kept trying to think of proofs of that. I kept proving it but then when I went to explain it to somebody the proof had a fallacy...*

Once Thurston began to imagine what a fiber in a fibered hyperbolic 3-manifold would have to look like geometrically—it would have to be very bent up—he came upon the pseudo-Anosov theory and he proved that a 3-manifold fibering over the circle has a hyperbolic structure if and only if the monodromy is homotopic to a pseudo-Anosov homeomorphism [78, 70].

When the first betti number of a fibered, hyperbolic 3-manifold M is greater than 1, it in fact fibers in infinitely many ways. Thurston gave a convenient way of organizing the elements of $H^1(M;\mathbb{Z})$ corresponding to fibers [80]. Specifically, he introduced a norm $\|\cdot\|$ on $H^1(M;\mathbb{Z})$, now called the Thurston norm, which essentially records the smallest genus representative of a given element of $H^1(M;\mathbb{Z})$. He proved that the unit ball is a finite-sided polyhedron and that the primitive elements of $H^1(M;\mathbb{Z})$ lying over a given open face are either all fibers or all not fibers.

By Thurston's theorem, each fiber α over a given fibered face has a pseudo-Anosov homeomorphism and hence a stretch factor $\lambda(\alpha)$ attached to it. We can extend $\log\lambda(\alpha)$ to a function on the rational points by homogeneity. Fried proved the function $\log\lambda(\alpha)\|\alpha\|$ is continuous and constant on rays and hence descends to a continuous function on each fibered face of the unit ball [31]; see also Long–Oertel [54]. Fried further proved this function is convex on the interior of each fibered face; Matsumoto proved strict convexity [59]. What is more, as we approach the boundary of the cone, $\log\lambda(\alpha)\|\alpha\|$ goes to infinity. Fried also proved that all of the fibers over a given fibered face are flow equivalent; in other words, there is a single flow in the circle direction so that the first return map for each fiber is the monodromy.

McMullen [61] gave a finer invariant on each fibered face F called the Teichmüller polynomial. This polynomial Θ_F lies in $\mathbb{Z}[H_1(M;\mathbb{Z})/\text{torsion}]$. If we write $\Theta_F = \sum a_h \cdot h$, then the stretch factor λ associated to any $\alpha \in H^1(M;\mathbb{Z})$ lying over F is the largest root of the polynomial $p(t) = \sum a_h \cdot t^{\alpha(h)}$. McMullen used this to give a new proof of the strict convexity. Notice that the strict convexity implies the existence of a unique minimum, which has been studied by Sun [75].

Thurston noticed that the set of stretch factors for pseudo-Anosov elements of $\mathrm{Mod}(S_g)$ is closed and discrete in $\mathbb{R}$; in particular it has a least element λ_g (see also [5, 45]). Penner showed that $\log\lambda_g$ tends to 0 at the rate $1/g$. [72]. McMullen observed that if we take a sequence α_i of fibers over a given fibered face whose Thurston norms go to infinity and whose projections to the unit sphere converge to the interior of the face, then by Fried's work the associated stretch factors have Penner's asymptotics [61].

The problem of understanding pseudo-Anosov maps with small stretch factor has received much attention recently [1, 8, 17, 25, 36, 42, 43, 51, 52, 53, 64, 82], although λ_g is only known explicitly for $g = 1, 2$; see [20]. Farb, Leininger, and the reviewer proved that there is a finite set of cusped hyperbolic 3-manifolds with the property that each of the infinitely many smallest-stretch pseudo-Anosov maps is (after deleting singular points) the monodromy of a fiber of one of these manifolds [26]; see also [4].

Today, we know that fibered hyperbolic 3-manifolds play a central role in the world of 3-manifolds: Wise proved that every cusped hyperbolic 3-manifold has a finite cover that is fibered [85], and Agol has proven this for closed hyperbolic 3-manifolds [3]. These theorems, together with the proof of Thurston's geometrization conjecture by Perelman, put the final exclamation point on Thurston's visionary program to understand 3-manifolds by their geometric pieces [79].

Dynamics. A number of the basic dynamical facts about individual pseudo-Anosov homeomorphisms were discovered right away:

(1) A pseudo-Anosov homeomorphism is topologically transitive (that is, it has a dense orbit).
(2) The periodic points for a pseudo-Anosov homeomorphism are dense.
(3) The foliations for a pseudo-Anosov homeomorphism are uniquely ergodic (i.e., each admits a unique transverse measure up to scale).
(4) A pseudo-Anosov homeomorphism has the minimal entropy among all homeomorphisms in its homotopy class.
(5) A pseudo-Anosov homeomorphism has the minimum number of periodic points for each period in its homotopy class.
(6) Any two homotopic pseudo-Anosov homeomorphisms are conjugate by a diffeomorphism.
(7) As a dynamical system, a pseudo-Anosov homeomorphism is isomorphic (in the measure-theoretic sense) to a Bernoulli shift.

Many of these facts are proven using the theory of Markov partitions, another cousin of foliations, laminations, train tracks, and geodesic currents. All of the proofs are contained in the book under review except (5), which can be found in [6, 12, 27, 37].

One lower bound for the number of fixed points of a self-map of a space is the Nielsen number. Ivanov proved that every mapping class has a representative where the number of fixed points is equal to the Nielsen number [44]. Fried showed there are pseudo-Anosov maps with no periodic points of period less than any prescribed n [31]. Handel proved that a pseudo-Anosov map has the minimal number of orbits in its homotopy class [37].

There are two ways to extend the pseudo-Anosov theory to noncompact surfaces of infinite type. Franks and Handel proved that a diffeomorphism of a surface has a normal form when restricted to the complement of the fixed points; if the set of fixed points is finite, this is just the Thurston normal form [30]. Cantwell–Conlon, Fenley, and Handel–Miller classified end-periodic homeomorphisms of noncompact surfaces [18, 29, 39].

A pseudo-Anosov homeomorphism is not differentiable at its singular points. However, Gerber and Katok showed that it is topologically conjugate—via a homeomorphism isotopic to the identity—to a diffeomorphism that is Bernoulli with respect to a smooth measure [35].

The braid group B_n is the mapping class group of a disk with n marked points. Boyland [15] introduced an ordering on the conjugacy classes in $\cup_n B_n$. Say that the conjugacy class C forces a conjugacy class C' if for every representative of a mapping class $f \in C$, we can erase the marked points for f and mark the points of some other finite orbit in order to obtain a representative for C'. This gives a partial ordering on conjugacy classes analogous to the Sharkovskii order on $\mathbb{N}$. In fact, Kolev has shown that if a pseudo-Anosov homeomorphism of a disk has a period 3 orbit, then it has an orbit of every period [49]; see also [34, 22, 38]. In other words, there are elements of B_3 that force elements of B_n for all $n \geq 3$. The higher genus version of this phenomenon is studied by Los [55].

Finally, we mention two facts about the dynamics of the action of the mapping class group on the Thurston boundary of Teichmüller space. It is elementary to see that the action is minimal: the orbit of each point is dense. Masur proved that the action is also ergodic [58].

Other directions. There are many, many aspects of mapping class groups and pseudo-Anosov maps that we have not broached here: Teichmüller and Weil–Petersson geometries on Teichmüller space, interval exchange transformations, the complex of curves and the large-scale geometric properties of the mapping class group, topological quantum field theories, cohomology of the mapping class group and surface bundles, as well as the deep and growing analogy with the outer automorphism group of a free group, to name just a few. Some places for the interested reader to start investigating are the book by Birman [11], the book and survey by Ivanov [47, 46], the survey by Harer [41], the textbook by the reviewer and Farb [27], the book of problems edited by Farb [23], and, last but not least, the research announcement by Thurston [81], which has an extensive list of references.

2. Travaux de Thurston and Thurston's work

Like most great books, Travaux de Thurston sur les Surfaces is known by its nickname: FLP. The greatest testament to FLP's importance is that it currently has 370 references on MathSciNet and, 34 years after its initial publication, the *first derivative* of this number is still increasing. There has been an explosion in research on mapping class groups, and FLP is still the best reference for the details about the Thurston compactification of Teichmüller space and Thurston's original proof of the classification.

FLP has fifteen "exposés," and is roughly divided into three parts. The first five exposés serve as introduction and background. Exposés 6–9 and 11 contain the heart of the book: coordinates on the space of measured foliations, proof that Teichmüller space is a ball, the Thurston compactification of Teichmüller space, the proof of the classification theorem for closed surfaces, and the classification theorem for surfaces with boundary. The other five exposés treat a variety of topics related to the theory of pseudo-Anosov homeomorphisms: their dynamics, their constructions, their uniqueness properties, and their associated fibered 3-manifolds. The last exposé is somewhat of an outlier; it presents Hatcher and Thurston's proof that the mapping class group is finitely presented.

The book really has five authors: Fathi, Laudenbach, Poénàru, Shub, and Fried, and two of the lectures were given by Douady and Marin. Despite this and despite the technical nature of the material, FLP is remarkably cohesive, detailed, and clear.

The recently published translation, Thurston's Work on Surfaces, is for the most part a faithful rendition of the original text. However, the translators have taken the liberty of making some changes, for instance to clarify a passage, insert a definition, correct a typo, modernize the language, or elucidate the big picture. One cosmetic change is worth noting: "arc jaune" was translated as "pants seam," probably to the lament of a generation of topologists. Overall, the translation is a most welcome addition for those who use FLP as a reference and are more comfortable with English than French. It is the reviewer's hope that this new version will also introduce Thurston's brilliant insights and imagination to even wider audiences and help inspire the present and future generations to pick up where he left off.

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