# HOMOMORPHISMS BETWEEN BRAID GROUPS

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Abstract. We give a complete classification of homomorphisms from the braid group on $n$ strands to the braid group on $2n$ strands when $n$ is at least 5. We also classify endomorphisms of the braid group on 4 strands, as well as homomorphisms from the commutator subgroup of the braid group on $n$ strands to the braid group on $2n-5$ strands. Our classifications suggest a recursive classification of homomorphisms between any braid groups. We also give a simple, geometric proof of a theorem of Lin that highly constrains the holomorphic maps that may exist between spaces of monic, square-free polynomials of two given degrees.

## 1. Introduction

Let $B_n$ denote the braid group on $n$ strands. A fundamental problem about these groups is to classify all homomorphisms $B_n \to B_m$ for various $n$ and $m$. Work of Artin [4], Lin [32], Dyer–Grossman [16], Bell–Margalit [6], and Castel [11] gives a complete classification for $n \geq 6$ and $m \leq n+1$. We extend the classification to the case $n \geq 4$ and $m \leq 2n$. For $m = 2n$ there are three new types of homomorphisms that do not arise when $m < 2n$.

As usual we denote the standard generators for $B_n$ by $\sigma_1, \ldots, \sigma_{n-1}$. We have the following *standard homomorphisms* $B_n \to B_{2n}$ (we compose braids right to left).

(1) *Trivial:* $\sigma_i \mapsto 1$

(2) *Inclusion:* $\sigma_i \mapsto \sigma_i$

(3) *Diagonal inclusion:* $\sigma_i \mapsto \sigma_i \sigma_{n+i}$

(4) *Flip diagonal inclusion:* $\sigma_i \mapsto \sigma_i \sigma_{n+i}^{-1}$

(5) *$k$-twist cabling:* $\sigma_i \mapsto \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i}\sigma_{2i-1}^{k}$

Note that there is one $k$-twist cabling map for each $k \in \mathbb{Z}$. Figure 1 shows the image of $\sigma_i$ under the $k$-twist cabling map. The first two maps also define homomorphisms $B_n \to B_m$ with $m < 2n$.

Let $G$ and $H$ be groups, and let $\rho : G \to H$ be a homomorphism. Further let $L : G \to \mathbb{Z}$ be a homomorphism, and let $t \in H$ be an element of the centralizer of $\rho(G)$. The *transvection* of $\rho$ by $t$ is the homomorphism

$$\rho^t(g) = \rho(g)t^{L(g)}.$$

There is an equivalence relation on the set of homomorphisms $G \to H$ where

$$\rho_1 \sim \rho_2 \iff \rho_2 = \tau \circ \rho_1^t$$

where $t$ lies in the centralizer of $\rho_1(G)$ and $\tau$ is an automorphism of $H$. We remark that a homomorphism $G \to H$ is equivalent to the trivial map if and only if it has cyclic image.

**Theorem 1.1.** *Let $n \geq 5$. Any homomorphism $\rho : B_n \to B_{2n}$ is equivalent to exactly one of the standard homomorphisms.*

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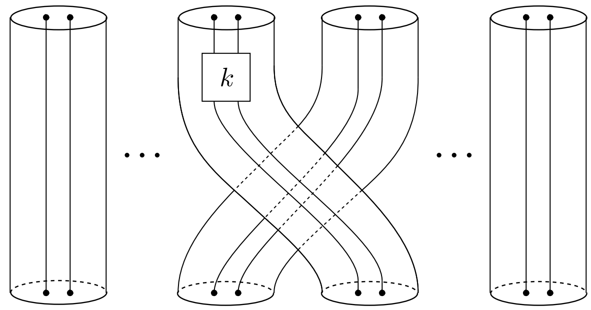


FIGURE 1. The image of $\sigma_i$ under the $k$-twist cabling map: the $i$th and $(i+1)$st cable cross, and inside the $i$th cable there are $k$ half-twists

Of course Theorem 1.1 is only as useful as our understanding of $\mathrm{Aut}(B_n)$ and the transvections. It is a theorem of Dyer–Grossman that $\mathrm{Aut}(B_n)$ is isomorphic to $B_n/Z(B_n) \rtimes \mathbb{Z}/2$, where the first factor acts by conjugation and the second factor acts by inversion: $\sigma_i \mapsto \sigma_i^{-1}$.

As for the transvections, the map $L : B_n \to \mathbb{Z}$ can be taken to be the abelianization, which maps $\sigma_i$ to 1 for all $i$. Also, the centralizers of the images of the standard homomorphisms $B_n \to B_{2n}$ are completely understood. We give a complete description in Section 10.

*Maps to smaller braid groups.* Theorem 1.1 further gives a classification of all homomorphisms $B_n \to B_m$ with $m < 2n$. Indeed, given a map $\rho : B_n \to B_m$ with $m \leq 2n$, we may post-compose with the inclusion $B_m \to B_{2n}$ to obtain a homomorphism to which Theorem 1.1 applies. We say that $\rho$ is standard if the composition is.

**Corollary 1.2.** *Let $n \geq 5$ and let $n \leq m < 2n$. Any homomorphism $\rho : B_n \to B_m$ is equivalent to a standard homomorphism, that is, the trivial map or the inclusion map.*

*Sharpness of the lower bound.* The lower bound $n \geq 5$ in the statement of Theorem 1.1 is sharp. Indeed, there is a surjective homomorphism $B_4 \to B_3$ given by $\sigma_1, \sigma_3 \mapsto \sigma_1$ and $\sigma_2 \mapsto \sigma_2$; we refer to this as the standard homomorphism $B_4 \to B_3$. Also, there are many non-standard homomorphisms $B_3 \to B_n$ for $n \geq 3$, since there exist pairs of pseudo-Anosov mapping classes that satisfy the braid relation. Combining these maps we obtain for any $n \geq 3$ homomorphisms $B_3 \to B_n$ and $B_4 \to B_n$ that are not equivalent to any standard homomorphism.

Despite the existence of the exceptional homomorphism $B_4 \to B_3$, we are still able to completely classify all homomorphisms $B_4 \to B_4$ (hence all homomorphisms $B_4 \to B_3$) by extending our methods to this case; see Theorem 7.5 in Section 8.

*Homomorphisms of the commutator subgroup.* The commutator subgroup $B_n'$ of $B_n$ is the kernel of the so-called length homomorphism $B_n \to \mathbb{Z}$ given by $\sigma_i \mapsto 1$ for all $i$. There is an inclusion map $i : B_{n-2} \to B_n'$ given by $\sigma_i \mapsto \sigma_i \sigma_{n-1}^{-1}$ for $i \in \{1, \ldots, n-3\}$. A homomorphism $\rho : B_n' \to B_{2n-5}$ thus induces a homomorphism $\rho \circ i : B_{n-2} \to B_{2n-5}$. By Theorem 1.1, the latter is equivalent to the trivial map or the inclusion map; this gives a strong restriction on $\rho$. Also, since $B_n'$ is perfect, all transvections of homomorphisms of $B_n$ are trivial. We thus arrive at the following corollary.

**Corollary 1.3.** *Let $n \geq 7$. If $\rho : B_n' \to B_{2n-5}$ is a nontrivial homomorphism, then there is an automorphism $\tau$ of $B_{2n-5}$ so that $\tau \circ \rho$ is equal to the inclusion map.*

We derive Corollary 1.3 from Theorem 1.1 in Section 12. This corollary generalizes a result of the second and third authors [28], who proved for $n \geq 7$ that any nontrivial homomorphism $B_n' \to B_n$ is equivalent to the inclusion map. The argument in that paper uses a different approach, the theory of totally symmetric sets. Orevkov [36] later extended the classification of homomorphisms $B_n' \to B_n$ for all $n \geq 1$.

In a 1996 preprint, Vladmir Lin asked a series of four increasingly general questions [31, 0.9.2(b)–0.9.2(e)] about endomorphisms of $B_n'$, the last being: does every automorphism of $B_n'$ extend to an automorphism of $B_n$? The theorem of the second and third authors already implies that the answer is yes, and Corollary 1.3 is a further extension.

Prior to all of these results, the group of automorphisms of $B_n'$ was determined for $n \geq 4$ by Orevkov [36]. An alternate proof for $n \geq 7$ was given by McLeay [34].

*Spaces of polynomials.* Let $\mathrm{Poly}_n$ denote the space of monic, square-free polynomials of degree $n$. This space is the same as the space of unordered configurations of $n$ points in the plane (the $n$ points are the roots). As such, we have that $\pi_1(\mathrm{Poly}_n)$ is isomorphic to $B_n$.

There is a surjective map $\mathrm{Poly}_4 \to \mathrm{Poly}_3$ that arises in the resolution of quartic polynomials into cubic polynomials. The induced map on fundamental groups is the surjective homomorphism $B_4 \to B_3$ described above.

By work of Lin [33, Theorem 9.4] and Murasugi [35], a holomorphic map $\mathrm{Poly}_n \to \mathrm{Poly}_m$ induces a homomorphism $B_n \to B_m$ that sends periodic elements to periodic elements (a braid is periodic if and only if it has a central power). Lin refers to such a homomorphism as a *special homomorphism*.

Lin proved the following result [32, Corollary 1.16].

**Theorem 1.4** (Lin)**.** *Let $n \geq 5$. If $n(n-1)$ does not divide $m(m-1)$ then every special homomorphism $B_n \to B_m$ has cyclic image.*

We give a simple, geometric proof of Theorem 1.4 in Section 4; see Proposition 4.1(1) and Lemma 7.2.

Our results give further information about special homomorphisms (hence about maps $\mathrm{Poly}_n \to \mathrm{Poly}_m$). For instance, Theorem 1.1 implies for $n \geq 5$ that the only special homomorphisms $B_n \to B_n$ are equivalent to the identity or the trivial map (this part follows from Castel's work; see below).

Also, our classification of homomorphisms $B_4 \to B_4$ (Theorem 7.5) shows that the only special homomorphisms $B_4 \to B_4$ are equivalent to the trivial map, the identity map, or the composition $B_4 \to B_3 \to B_4$, where the first map is the standard surjective homomorphism and the second map is the standard inclusion. Finally, it follows from Theorem 7.5 that the only special homomorphisms $B_4 \to B_3$ are equivalent to either the trivial map or to the standard map. We can interpret the last statement as saying that there is essentially only one way to resolve quartic polynomials into cubic polynomials.

*Maps to larger braid groups.* Because the number of types of standard homomorphisms $B_n \to B_m$ jumps from 2 to 5 when $m$ increases from $2n-1$ to $2n$, it may seem hopeless to classify all homomorphism $B_n \to B_m$ when $m$ is large. However, there is a concrete sense in which the homomorphisms $B_n \to B_{2n}$ are all built from homomorphisms $B_n \to B_n$, as we now explain.

In this discussion it will be useful to regard $B_n$ as the mapping class group of a disk $\mathbb{D}_n$ with $n$ marked points; this is the group of connected components of the group of homeomorphisms of $\mathbb{D}_n$ that fix the boundary pointwise. A map $\rho : B_n \to B_m$ is reducible if there is a multicurve $M$ in $\mathbb{D}_m$ preserved by $\rho(B_n)$. We should think of a reducible homomorphism as a type of cabling map, and in fact we will refer to them as such in what follows. In the standard braid picture, the multicurve $M$ traces out the boundary of a cable. It is possible that there is only one cable, for instance in the case of the inclusion map $B_n \to B_{n+1}$.

Our Theorem 1.1 implies in particular that every homomorphism $\rho : B_n \to B_m$ with $m \leq 2n$ is equivalent to a cabling. For example, the inclusion map has one cable with $n$ strands inside, the diagonal and flip diagonal inclusions have two cables, each with $n$ strands inside, and the $k$-twist cablings have $n$ cables, each with two strands inside.

A cabling map can have several levels of cables: cables within cables, within cables, etc. As explained in Section 6, a cabling $\rho : B_n \to B_m$ decomposes into a collection of homomorphisms from $B_n$ (or a subgroup of finite index) to braids groups $B_{m_i}$ with $m_i < m$. These homomorphisms describe the cabling at all the levels. Assuming homomorphisms $B_n \to B_{m_i}$ are already classified (by induction), we can obtain in this way a description of $\rho$ in terms of known homomorphisms. For example, the flip diagonal inclusion map $B_n \to B_{2n}$ decomposes into three maps: the inclusion map $B_n \to B_n$, the inversion map $B_n \to B_n$, and the trivial map $B_n \to B_2$ (the latter is the exterior component, or outer level, of the flip diagonal inclusion map, described in Section 6).

With this in mind, we may give a recursive definition of what it means for a homomorphism $B_n \to B_m$ to be standard. We begin the recursive definition by starting with the base cases $m \leq n$. In the base cases, we say that a homomorphism $B_n \to B_m$ is standard if it is trivial, the identity, or the inversion map. Then for $m > n$ we say that a homomorphism $\rho : B_n \to B_m$ is standard if it is a cabling, and all of the induced homomorphisms to smaller braid groups are standard (or restrictions of standard homomorphisms). All of the standard homomorphisms $B_n \to B_{2n}$ defined at the start of the paper are standard in this sense.

**Question 1.5.** *Is it true that any homomorphism $\rho : B_n \to B_m$ is equivalent to a standard homomorphism, in the sense of our recursive definition?*

We expect that the answer to Question 1.5 is 'yes.' To prove this, it is enough to show that each $B_n \to B_m$ with $m > n$ is a cabling. The affirmative answer to the question would then follow by induction. Even if Question 1.5 were answered, though, it would not give an explicit classification of homomorphisms as in Theorem 1.1.

Question 1.5 is a sharpening of the question of whether all homomorphisms between mapping class groups are induced—in Mirzakhani's words—from "some manipulations of surfaces" [3, p. 3]. There are many results showing that homomorphisms between mapping class groups are induced by inclusions of surfaces; see the work of Ivanov [23], Irmak [20, 21], Shackleton [37], and Aramayona–Souto [3]. There are many other related results, most of which can be found in the references to Ivanov's paper on MathSciNet.

Aramayona–Leininger–Souto [2] gave examples of injective homomorphisms between mapping class groups of closed surfaces. Since there are no inclusions between closed surfaces, these homomorphisms are of a different nature than the ones discussed in the previous paragraph; in fact, there are pseudo-Anosov mapping classes that map to multitwists. On the other hand, the Aramayona–Leininger–Souto homomorphisms are constructed by lifting to a covering space, which is indeed a manipulation of surfaces.

On might expect that the Aramayona–Leininger–Souto construction could be used to produce homomorphisms $B_n \to B_m$ where pseudo-Anosov elements map to multitwists. This does not seem to be the case, since the typical branched cover of $\mathbb{D}_n$ has positive genus.

*Prior results.* An early precursor to Theorem 1.1 is a theorem of Artin from 1947, which states that any homomorphism from $B_n$ to the symmetric group $S_n$ with transitive image is either cyclic or is conjugate to the standard projection [4]. In 1996, Lin [32, Theorem F] classified all homomorphisms $B_n \to S_{2n}$ with transitive image (they all arise from our standard homomorphisms $B_n \to B_{2n}$). Using the classification of homomorphisms $B_n \to S_k$ with $k < n$, he further proved [32, Theorem A] that every homomorphism $B_n \to B_k$ has cyclic image provided $k < n$.

Dyer–Grossman [16, Theorem 19] proved in 1981 that $\mathrm{Aut}(B_n) \cong B_n/Z(B_n) \rtimes \mathbb{Z}/2$, answering a question of Artin (Pietrowski and Solitar [25] previously proved this for $n \leq 4$ and conjectured the more general statement). Bell and the third author proved in 2006 that every injective homomorphism $B_n \to B_{n+1}$ is a transvection of a conjugate of the standard inclusion [6, Main Theorem 2]. Castel improved on this result by showing that every homomorphism $B_n \to B_{n+1}$ with non-cyclic image is of the same form [11, Theorem 4(iii)].

There are many other works describing various types of maps between various types of braid groups; see, for instance, the work of An [1], Bardakov [5], Bell and the third author [7], Childers [14], Cohen [15], Irmak–Ivanov–McCarthy [22], Leininger and the third author [30], McLeay [34], Orevkov [36], Zhang [39] and the first author [12].

*Idea of the proof.* The overarching strategy for the proof of Theorem 1.1 is to classify homomorphisms $B_n \to B_m$ inductively. The base case $m = n$ is due to Castel (as mentioned above, this is [11, Theorem 4(iii)]). We give a new, relatively short proof of Castel's theorem. Also, we improve the hypothesis of Castel's theorem from $n \geq 6$ to $n \geq 5$. As in the statement of Theorem 1.1, this bound is sharp because of the surjective homomorphism $B_4 \to B_3$.

For both the base case and the inductive step, we analyze a homomorphism $B_n \to B_m$ by considering the various possibilities for the images of the periodic elements of $B_n$. The braid group is torsion free; the periodic elements of $B_n$ are defined to be the ones that are periodic in the sense of the Nielsen–Thurston classification.

In the arguments, we focus on a specific periodic element $\alpha_1 \in B_n$; this corresponds to rotating $\mathbb{D}_n$ by one "click." According to the Nielsen–Thurston classification theorem, there are three possibilities for the image of $\alpha_1$ under a homomorphism $\rho : B_n \to B_m$: it can be either pseudo-Anosov, periodic, or reducible. In the pseudo-Anosov and reducible cases, we show that $\rho$ has cyclic image, and in the periodic case we show that either $\rho$ has cyclic image or it is equivalent to the identity. We give a more detailed description of our approach in Section 2.

Our approach in this paper stands in contrast to the works of Ivanov [23], Ivanov–McCarthy [24], Bell and the third author [6], and Aramayona–Souto [3], where homomorphisms between mapping class groups are understood by considering the different possibilities for the image of a Dehn twist. The approach of using periodic elements has, on the other hand, also been used in the theory of mapping class groups, for instance in the work of Harvey–Korkmaz [19], of Tchangang [38], and of Lanier and the first author [13].

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## 2. Setup and Overview

We give here an overview of the paper, and also introduce some notation and ideas that will be used throughout. In this paper we make extensive use of the Nielsen–Thurston classification for mapping class groups and the related theory of canonical reduction systems; see [18, Chapter 13].

*Outline.* As per the introduction, we prove Theorem 1.1 by classifying maps $B_n \to B_m$ inductively. Along the way, we work with the group of even braids $B_n^2$. This is the same as the subgroup of $B_n$ generated by all squares of all elements, and also the same as the kernel of the mod 2 abelianization homomorphism $B_n \to \mathbb{Z}/2$. Our induction has three steps.

*Base case.* A classification of homomorphisms $B_n \to B_n$.

*Extension of the base case.* A classification of homomorphisms $B_n^2 \to B_n$.

*Inductive step.* A classification of homomorphisms $B_n \to B_m$ with $n < m \leq 2n$.

As above, the base case is a theorem of Castel; we give our new proof in Section 7. We complete the other two steps in Sections 8 and 11, respectively. Our proof of Castel's theorem and the extension to the even braid group require several tools, which we build in Sections 3 through 6. All of the tools, except for the ones in Section 3 are used in the inductive step. We introduce a further tools for the main theorem—a classification of certain cabling maps—in Section 9. In Section 10 we prove that the standard homomorphisms $B_n \to B_{2n}$ are pairwise inequivalent; along the way, we give in Section 10 a classification of the transvections of the standard homomorphisms.

As discussed in the introduction, for all three steps we analyze a homomorphism from $B_n$ to $B_m$ by considering the various possibilities for the images of the periodic elements of $B_n$. An element is periodic if it is periodic in the sense of the Nielsen–Thurston classification. Equivalently, an element of $B_n$ is periodic if its image has finite order in $\bar{B}_n$, the quotient of $B_n$ by its center $Z(B_n)$.

The quotient $\bar{B}_n$ is isomorphic to a subgroup of the mapping class group of a sphere $S_{0,n+1}$ with $n+1$ marked points, namely, the subgroup consisting of elements that fix a distinguished marked point $p$. Given a homomorphism $\rho : B_n \to B_m$ we will often consider the associated homomorphism $\bar{\rho} : B_n \to \bar{B}_m$, which is the post-composition of $\rho$ with the projection $B_m \to \bar{B}_m$.

We specifically utilize two particular periodic elements $\alpha_1 = \sigma_{n-1} \cdots \sigma_2 \sigma_1$ and $\alpha_2 = \sigma_{n-1} \cdots \sigma_2 \sigma_1^2$ (again composing braids right to left). We have that $\alpha_1^n$ is equal to $\alpha_2^{n-1}$; we denote this element by $z$. The element $z$ generates $Z(B_n) \cong \mathbb{Z}$. We denote by $\bar{\alpha}_k$ the image of $\alpha_k$ in $\bar{B}_n$, so $\bar{\alpha}_1$ is a rotation of $S_{0,n+1}$ by $2\pi/n$ and $\bar{\alpha}_2$ is a rotation by $2\pi/(n-1)$.

*Base case.* Let us first discuss the proof of Castel's theorem. Let $\rho : B_n \to B_n$ be a homomorphism and let $\bar{\rho}$ be the associated homomorphism to $B_n \to \bar{B}_n$. As in the introduction, there are three possibilities for $\bar{\rho}(\alpha_1)$ under a homomorphism $\rho : B_n \to B_m$: it can be either pseudo-Anosov, periodic, or reducible.

When $\bar{\rho}(\alpha_1)$ is pseudo-Anosov, we show that $\rho$ has cyclic image. The idea is that if $\bar{\rho}(\alpha_1)$ is pseudo-Anosov, then its power $\bar{\rho}(z)$ is also pseudo-Anosov. Since $\bar{\rho}(B_n)$ is contained in the centralizer of $\bar{\rho}(z)$, and since the centralizers of pseudo-Anosov mapping classes are completely understood (by work of McCarthy), we can conclude that $\bar{\rho}$ has abelian (hence cyclic) image, and hence $\rho$ has cyclic image. The details of this argument are given in Section 5.

When $\bar{\rho}(\alpha_1)$ is periodic we show that $\rho$ either has cyclic image or is equivalent to the identity homomorphism. The argument proceeds as follows. We show in Section 4 that if $\bar{\rho}(\alpha_1)$ is periodic and $\rho$ has non-cyclic image then (up to replacing $\rho$ by an equivalent homomorphism) $\bar{\rho}(\alpha_1)$ generates $\langle \bar{\alpha}_1 \rangle$. To give the idea, here is why $\bar{\rho}(\alpha_1)$ may not equal $\bar{\alpha}_2$. In this case,

$$\bar{\rho}(\alpha_1^{n-1}) = \bar{\alpha}_2^{n-1} = 1.$$

But the the normal closure of $\alpha_1^{n-1}$ in $B_n$ contains the commutator subgroup (this fact is an instance of the well-suited curve criterion of Lanier and the third author [29]). It follows that $\bar{\rho}$, hence $\rho$, has cyclic image.

Then, assuming that $\rho$ has non-cyclic image, we show that $\bar{\rho}(\alpha_1)$ is equal to $\bar{\alpha}_1^{\pm 1}$. To do this we consider the interaction between $\bar{\rho}(\alpha_1)$ and the canonical reduction system $M$ of $\bar{\rho}(\sigma_1)$. Since $\alpha_1^k \sigma_1 \alpha_1^{-k}$ commutes with $\sigma_1$ for $2 \leq k \leq n-1$, it must be that the image of $M$ under $\bar{\rho}(\alpha_1)^k$ is disjoint from $M$ for such $k$. We prove a general lemma in Section 3 that gives strong restrictions for a multicurve to be disjoint from its image under a rotation or a set of rotations, and use this to prove that $\bar{\rho}(\alpha_1) = \bar{\alpha}_1^{\pm 1}$. We then leverage the fact that $\bar{\rho}$ agrees on $\alpha_1$ with the projection $B_n \to \bar{B}_n$ in order to conclude that $\rho$ is equivalent to the identity homomorphism.

If $\rho(\alpha_1)$—equivalently, $\bar{\rho}(\alpha_1)$—is reducible, this means that $\rho(\alpha_1)$ preserves a multicurve $M$ in $\mathbb{D}_n$. It follows that $\rho(z)$, hence all of $\rho(B_n)$, preserves $M$. It must be that $M$ has fewer than $n$ components. We may then apply a theorem of Lin, which says that any homomorphism from the commutator subgroup $B_n'$ to $S_k$ with $k < n$ is trivial. This implies that $B_n'$ fixes each component of $M$. We prove in Section 6 that the subgroup of $B_n$ consisting of elements that fix each component of $M$ is isomorphic to a semi-direct product of a braid group with a direct product of braid groups. The associated braids all have fewer than $n$ strands, and so we may apply induction in order so show that $\rho$ has cyclic image.

After proving (our extension of) Castel's theorem we give the classification of homomorphisms $B_4 \to B_4$ at the end of Section 8. The proof follows the same outline as the one we use for Castel's theorem.

*Extension of the base case.* The proof of the extension of Castel's theorem to the case of $B_n^2$ follows the same outline. Since $\alpha_1$ does not lie in $B_n^2$ when $n$ is even and since $\sigma_1$ never lies in $B_n^2$, we must also consider the elements $\alpha_2$ and $\sigma_1^2$. With these minor modifications, the proof of Castel's theorem applies almost verbatim. We remark that it is possible to derive Castel's theorem from the corresponding theorem for $B_n^2$; the main reason we prove Castel's theorem first is that it is of more independent interest.

*Inductive step.* The inductive step again follows the same approach as the proof of Castel's theorem. Given $\rho : B_n \to B_m$ with $n < m \leq 2n$ we consider the three possibilities for $\bar{\rho}(\alpha_1)$. If $\bar{\rho}(\alpha_1)$ is pseudo-Anosov or periodic then as before we conclude that $\rho$ has cyclic image. In fact this case is easier than the case $m = n$ since it is not possible for $\bar{\rho}(\alpha_1)$ to be a power of $\bar{\alpha}_1$.

The case where $\bar{\rho}(\alpha_1)$ is reducible is the most subtle part of the proof. In this case we conclude as before that $\rho(B_n)$ preserves a multicurve $M$. If $M$ consists of $n$ curves surrounding two marked points each and $B_n$, through $\rho$, acts standardly on the set of components of $M$, then $m = 2n$ and $\rho$ is what we call a 2-fold cabling map, and so we may apply our classification of 2-fold cabling maps $B_n \to B_{2n}$ from Section 9.

Another case is where $M$ consists of a single curve surrounding more than $n$ (and necessarily fewer than $m$) marked points; in this case we may restrict to the interior of $M$ and apply induction. The most difficult case, which only arises when $m = 2n$, is where $M$ consists of two curves surrounding $n$ marked points each and where $\rho(B_n)$ acts nontrivially on the components of $M$; in this case $B_n^2$ acts trivially on the components of $M$ and so we may apply our extension of the base case in order to show that $\rho$ is a diagonal embedding.

## 3. Rotations and curves

Recall that $\alpha_1$ and $\alpha_2$ are the periodic elements of $B_n$ corresponding to rotations of $\mathbb{D}_n$ by $2\pi/n$ and $2\pi/(n-1)$, respectively, and that $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are the images in $\bar{B}_n$. In this section we prove two facts about the interactions between the rotations $\bar{\alpha}_k$ and simple closed curves in $S_{0,n+1}$. First, in Section 3.1 we prove Proposition 3.1, which states that any curve $c$ in $S_{0,n+1}$ must intersect its image under $\bar{\alpha}_k$ for $k \in \{1, 2\}$. Then in Section 3.2 we prove Proposition 3.3, which states that if a curve $c$ is disjoint from each $\bar{\alpha}_1^i(c)$ with $2 \leq i \leq n-1$ then $c$ surrounds at most two marked points (and similar for $\bar{\alpha}_2$).

3.1. **Curves and primitive rotations.** As discussed above, the goal of this section is to prove the following proposition.

**Proposition 3.1.** *Let $n \geq 3$, let $k \in \{1, 2\}$, let $\epsilon \in \{\pm 1\}$, and let $c$ be an essential simple closed curve in $S_{0,n+1}$. Then*

$$i(c, \bar{\alpha}_k^{\epsilon}(c)) \neq 0.$$

The basic idea of the proof of Proposition 3.1 is that if we had $i(c, \bar{\alpha}_k(c)) = 0$, then we could find an arc $\delta$ in $S_{0,n+1}$ that is disjoint from its image $\bar{\alpha}_k(\delta)$ (take $\delta$ to be any arc in the interior of $c$, that is, the complementary component not containing the distinguished marked point $p$). We would like to show that this is impossible. The following lemma proves a version of the desired statement for arcs, with the mapping class $\bar{\alpha}_k$ represented by a Euclidean rotation.

We first require some setup that will be used throughout this section. Let $R_1$ be the open unit disk in the Euclidean plane with $n$ marked points equally spaced on a circle centered at the origin, and let $R_2$ be open unit disk in the Euclidean plane with $n-1$ marked points equally spaced on a circle centered at the origin and with one additional marked point at the origin. Let $r_1$ denote the clockwise rotation of $R_1$ by $2\pi/n$ and let $r_2$ denote the clockwise rotation of $R_2$ by $2\pi/(n-1)$. Under appropriate identifications of $R_1$ and $R_2$ with $S_{0,n+1}$, the maps $r_1$ and $r_2$ correspond to $\bar{\alpha}_1$ and $\bar{\alpha}_2$, respectively.

Let $k \in \{1, 2\}$. By an *arc* in $R_k$ we mean the image of an embedding $[0, 1] \to R_k$ where the preimage of the set of marked points is $\{0, 1\}$. We say that an arc is *essential* if it is not homotopic (through arcs) to a marked point. When we say that two arcs in $R_k$ are disjoint, we mean this in the strictest possible sense: we mean that their interiors are disjoint and also that their endpoints are disjoint.

A half-open arc in $R_k$ is a proper embedding $[0, 1) \to R_k$ where the preimage of the set of marked points is $\{0\}$. Let $\Delta$ be a collection of $n-1$ disjoint half-open arcs that decompose $R_2$ into $n-1$ fundamental domains for the action of $r_2$; see Figure 2. We say that an arc $\delta$ lies in minimal position with $\Delta$ if it has the fewest number of intersections with $\Delta$ in its homotopy class. By the bigon criterion, this is equivalent to the statement that $\delta$ forms no bigons with $\Delta$. (It is possible to define a version of $\Delta$ for the action of $r_1$ on $R_1$, but such a $\Delta$ would not be a collection of arcs.)

**Lemma 3.2.** *Let $n \geq 3$, let $k \in \{1, 2\}$, and let $\delta$ be an essential arc in $R_2$ that lies in minimal position with $\Delta$. Then $r_2(\delta)$ and $\delta$ are not disjoint.*

*Proof.* To have a pair of completely disjoint arcs requires $n \geq 4$, so we assume $n \geq 4$. Assume for the sake of contradiction that $\delta$ is an arc in $R_2$ that lies in minimal position with $\Delta$ and is disjoint from $r_2(\delta)$. The collection of arcs $\Delta$ cuts $\delta$ into a sequence of sub-arcs $\delta_1, \ldots, \delta_\ell$. Each

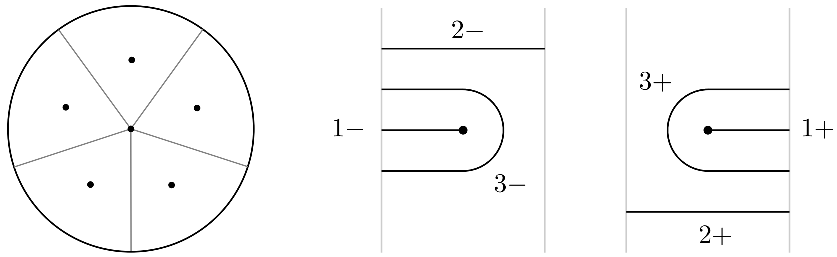


FIGURE 2. Left: the collection of arcs $\Delta$; Right: the 6 different types of arcs in a fundamental domain for $r_2$

$\delta_i$ lies in one fundamental domain (there may be more than one sub-arc in a fundamental domain).

Because $\delta$ and $\Delta$ do not form any bigons, there are only 6 possibilities for each $\delta_i$ up to homotopy, where homotopies keep the endpoints of $\delta_i$ on $\Delta$ but are allowed to move the endpoints along $\Delta$. The 6 possibilities are shown in the right-hand side of Figure 2 (we have distorted the fundamental domains there so that they look like rectangles instead of wedges). As in the figure, we refer to the 6 types of arcs as types 1+, 1−, 2+, 2−, 3+, and 3−. We further say that an arc is of type 1 if it is of type 1+ or 1−, etc.

Similarly, we may divide the geodesic $r_2(\delta)$ into sub-arcs along $\Delta$. We denote the sub-arcs by $r_2(\delta)_i$. They of course fall into the same 6 types as the sub-arcs of $\delta$. Specifically, since $\Delta$ is invariant under $r_2$, each $r_2(\delta_i)$ is the sub-arc $r_2(\delta)_i$, and hence each $r_2(\delta)_i$ has the same type as $\delta_i$.

There are exactly two sub-arcs $\delta_i$ of type 1, namely, $\delta_1$ and $\delta_\ell$. The proof proceeds by analyzing the possibilities for the type of $\delta_i$, beginning with $i = 1$ and proceeding inductively.

We may assume that $\delta_1$ is of type 1+. Indeed, if $\delta_1$ is of type 1−, then we may interchange $\delta$ and $r_2(\delta)$ and interchange the center point of $R_k$ with the exterior puncture in order to obtain the desired situation. We henceforth assume that $\delta_i$ and $r_2(\delta_1)$ are of type 1+.

The arc $\delta_2$ lies in the same fundamental domain as $r_2(\delta)_1$. Since $\delta \cap r_2(\delta) = \emptyset$ by assumption and since the marked points at the ends of $\delta$ and $r_2(\delta)$ are distinct, it follows that $\delta_2$ must be of type 2. We assume that it is of type 2+; the other case is essentially the same (alternatively, we may again interchange the center point of $R_k$ with the exterior puncture).

Again, $r_2(\delta)_2$ is of type 2+. Arguing as before, and using only the property that $\delta \cap r_2(\delta) = \emptyset$ we see that $\delta_3$ must also be of type 2+. Continuing in this way inductively, we conclude that each $\delta_i$ with $i > 1$ must be of type 2+. This contradicts the fact that $\delta_\ell$ is of type 1. □

*Proof of Proposition 3.1.* We begin with some setup. Let $R_k$ be the disk with marked points defined above, and let $R_k^\circ$ be the surface obtained from $R_k$ by removing the marked points; this surface is homeomorphic to a sphere with $n + 1$ punctures.

There is a hyperbolic metric on $R_k^\circ$ and a representative $r_k$ of $\bar{\alpha}_k$ that acts by isometries [18, 7.1]. We fix this metric once and for all and refer to it as the hyperbolic metric on $R_k^\circ$ (while the metric depends on $k$, there will be no confusion in what follows). By a theorem of Brouwer, de Kérékjartò, and Eilenberg [8, 17, 26], any two rotations of a (punctured) sphere through a given angle are conjugate in the group of homeomorphisms, and so we may assume that the $r_k$ given here is the the $r_k$ from Lemma 3.2.

Suppose now for the sake of contradiction that $i(c, \alpha_k^\epsilon(c)) = 0$. By identifying the complement of the marked points and boundary in $\mathbb{D}_n$ with $R_k^\circ$, it follows that there is a curve $\bar{c}$ in $R_k^\circ$ so that $i(\bar{c}, \bar{\alpha}_k^\epsilon(\bar{c})) = 0$.

Let $\gamma$ be the geodesic representative of $\bar{c}$ in the hyperbolic metric on $R_k^\circ$. Since $r_k$ is an isometry of this metric, and since geodesics minimize intersection within homotopy classes, it must be that $\gamma \cap r_k(\gamma) = \emptyset$.

Let $\delta$ be any essential arc contained in the interior of $\gamma$ (this exists because $\gamma$ is essential). Because $\gamma \cap r_k(\gamma) = \emptyset$, it follows the interior of $\gamma$ and its image under $r_k$ are disjoint. It follows that $r_k(\delta)$ is disjoint from $\delta$. For $k = 2$ this is impossible by Lemma 3.2.

In the case $k = 1$ it also follows from the equality $\gamma \cap r_1(\gamma) = \emptyset$ that $\delta$ does not pass through the origin. Therefore, we may regard $\gamma$ as an arc in $R_2^\circ$ (the one with $n+2$ punctures) with the property that $\gamma \cap r_2(\gamma) = \emptyset$. Again this is impossible by Lemma 3.2. □

3.2. **Rotations and multicurves.** We now proceed to the second and final result of the section. As above the interior of a curve in $S_{0,n+1}$ is the complementary component not containing the distinguished marked point $p$.

**Proposition 3.3.** *Let $n \geq 5$, and let $M$ be a multicurve in $S_{0,n+1}$. Suppose that either*

*(1) $n \geq 5$ and $i(M, \bar{\alpha}_1^i(M)) = 0$ for $2 \leq i \leq n-2$, or*
*(2) $n \geq 6$ and $i(M, \bar{\alpha}_2^i(M)) = 0$ for $2 \leq i \leq n-3$.*

*Then $M$ is a single curve with exactly two marked points in its interior.*

*Proof.* We first prove the proposition in the case of the first hypothesis (about $\alpha_1$), and then prove it under the second hypothesis (about $\alpha_2$).

Assume that $n \geq 5$ and $i(M, \bar{\alpha}_1^i(M)) = 0$ for $2 \leq i \leq n-2$. We first treat the case where $M$ is (a priori) equal to a single curve $c$. Let $N$ be the largest even number less than $n$. Consider the multiset of curves

$$X = \{c, \bar{\alpha}_1^2(c), \bar{\alpha}_1^4(c), \ldots, \bar{\alpha}_1^N(c)\}$$

where $N$ is the largest even number less than or equal to $n-2$. The elements of $X$ have trivial intersection pairwise. Indeed, $c$ has trivial intersection with each by assumption, and by applying $\bar{\alpha}_1^{-2}$ to the other curves simultaneously, we see that $\bar{\alpha}_1^2(c)$ is disjoint from the curves that come after it, etc.

We claim that the elements of $X$ are also pairwise distinct (in other words the multiset $X$ is a set). Suppose to the contrary that $\bar{\alpha}_1^p(c) = \bar{\alpha}_1^q(c)$ where $0 \leq p < q \leq N$. Applying $\bar{\alpha}_1^{-p}$ to both curves we obtain the equality $c = \bar{\alpha}_1^i(c)$ where $0 < i \leq N$ (here $i = q - p$). We first treat the case where $i \neq N$. Applying $\bar{\alpha}_1$ to both sides of the equality $c = \bar{\alpha}_1^i(c)$ we obtain the equality $\bar{\alpha}_1(c) = \bar{\alpha}_1^{i+1}(c)$. By the assumptions on $c$ and $i$, we have that $i(c, \bar{\alpha}_1^{i+1}(c)) = 0$, and so $i(c, \bar{\alpha}_1(c)) = 0$. Since $c$ is essential by assumption, this contradicts Proposition 3.1. The case where $i = N$ is treated in the same way, with $\bar{\alpha}_1$ replaced by $\bar{\alpha}_1^{-1}$. This completes the proof of the claim.

Since each curve of $X$ is the image of $c$ under some mapping class, they all surround the same number of marked points and moreover the corresponding sets of marked points are disjoint. Since $|X| = N/2 + 1 \geq n/3$ for $n \geq 4$ and since the curves are distinct it follows that each curve, in particular the curve $c$, surrounds at most 2 marked points. Because $c$ is essential, it surrounds exactly 2. This completes the proof in the special case where $M = c$.

We now prove the general case of the proposition. Suppose for the sake of contradiction that $M$ has two components $c_1$ and $c_2$ (and possibly others). We consider the corresponding multisets of curves $X_1$ and $X_2$, defined in the same way as $X$ above. As above, the elements of $X_i$ are pairwise disjoint and distinct for each $i$. We conclude as above that each element of each $X_i$ is a curve surrounding exactly two marked points.

We claim that the elements of the multiset $X_1 \cup X_2$ are pairwise distinct. Suppose to the contrary that, say, $\bar{\alpha}_1^u(c_1) = \bar{\alpha}_1^v(c_2)$; we may assume without loss of generality that $u < v$. As above we obtain from this the equality $c_1 = \bar{\alpha}_1^i(c_2)$ with $0 < i \leq N$ and then (since $n \geq 5$) the equality $\bar{\alpha}_1^{\pm 1}(c_1) = \bar{\alpha}_1^{i\pm 1}(c_2)$, where $i \pm 1$ is chosen to lie in $[2, n-2]$. By Proposition 3.1 we have $i(c_1, \bar{\alpha}_1^{\pm 1}(c_1)) \neq 0$, and so combining this with the previous equality we have $i(c_1, \bar{\alpha}_1^{i\pm 1}(c_2)) \neq 0$. It follows that $i(M, \bar{\alpha}_1^{i\pm 1}(M)) \neq 0$, contrary to the assumption. This completes the proof of the claim.

We complete the proof now in two cases, first for $n$ even and then for $n$ odd. Assume that $n$ is even. In this case we have that each element of $X_1$ surrounds exactly two marked points, that the elements of $X_1$ are pairwise distinct and disjoint, and that $|X_1| = n/2$. In other words, the elements of $X_1$ surround the $n$ marked points of $\mathbb{D}_n$ in pairs. We also have that the elements of $X_2$ surround two marked points each, and that they are distinct and disjoint from the elements of $X_1$. This is a contradiction. The case where $n$ is odd is essentially the same, except that $|X_1| = (n-1)/2$. This completes the proof assuming the first hypothesis.

Now assume that $n \geq 6$ and $i(M, \bar{\alpha}_2^i(M)) = 0$ for $2 \leq i \leq n-3$. Let $q$ be the marked point of $S_{0,n+1}$ that is fixed by $\bar{\alpha}_2$ and is not equal to $p$. It must be that $q$ does not lie in the interior of any component of $M$, for otherwise $M$ would not be disjoint from its image under any power of $\bar{\alpha}_2$. It must also be that each component of $M$ contains at least two marked points in its exterior for the same reason. Therefore, we may forget the marked point $q$ and we obtain a multicurve in $S_{0,n}$ satisfying the first hypothesis of the proposition. Since $q$ was not contained in the interior of any component of $M$, the proposition follows. □

## 4. Torsion to torsion

The goal of this section is to prove the following proposition. The first statement of the proposition is due to Lin [32, Corollary 1.16]. We give here a simple, geometric proof.

For the statement of the proposition, recall that $z$ is the positive generator for $Z(B_n)$ and that $\bar{\rho} : B_n \to \bar{B}_m$ is the homomorphism associated to a given homomorphism $\rho : B_n \to B_m$.

**Proposition 4.1.** *Let $n \geq 5$, let $m \geq 1$, and let $\rho : B_n \to B_m$ be a homomorphism. Assume that $\bar{\rho}(z)$ is periodic.*

1. *If $n(n-1)$ does not divide $m(m-1)$, then $\bar{\rho}$ has cyclic image.*
2. *If $m = n$ then, up to post-composing $\rho$ by an inner automorphism of $B_m$, we either have that $\bar{\rho}$ has cyclic image or that $\bar{\rho}(\alpha_1) = \bar{\alpha}_1^k$ where $\gcd(k, n) = 1$.*

Note that in the statement of Proposition 4.1, the element $\alpha_1$ lies in $B_n$ and the element $\bar{\alpha}_1$ lies in $\bar{B}_m$. In what follows, when we refer to an element $\alpha_k$ or $\bar{\alpha}_k$, it will be clear from context which group it lies in.

In order to prove Proposition 4.1 we require the following lemma, which is a version of the well-suited curve criterion of Lanier and the second author of this paper [29]. In what follows, we denote by $H_c$ the element of $B_n$ given by a half-twist about a homotopy class of arcs $c$ in $\mathbb{D}_n$.

**Lemma 4.2.** *Let $n \geq 5$ and let $f \in B_n$. Suppose there is a homotopy class of arcs $c$ in $\mathbb{D}_n$ such that either*

1. *$c$ and $f(c)$ have disjoint representatives, or*
2. *$c$ and $f(c)$ have representatives that share one endpoint and have disjoint interiors.*

*Then the normal closure of $f$ in $B_n$ contains $B_n'$.*

*Proof.* The hypotheses imply that $H_c H_{f(c)}^{-1}$ is conjugate in $B_n$ to either $\sigma_1\sigma_2^{-1}$ or $\sigma_1\sigma_3^{-1}$. Since $H_c H_{f(c)}^{-1}$ is equal to $H_c f H_c^{-1} f^{-1}$ it follows that $\sigma_1\sigma_2^{-1}$ or $\sigma_1\sigma_3^{-1}$ lies in the normal closure of $f$. Since $\sigma_1\sigma_2^{-1}$ and $\sigma_1\sigma_3^{-1}$ are both normal generators for $B_n'$ (see [32, Remark 1.10]), the lemma follows. □

*Proof of Proposition 4.1.* We begin with the first statement. Assume that $n(n-1)$ does not divide $m(m-1)$.

We claim first that $\alpha_1^{m(m-1)}$ and $\alpha_2^{m(m-1)}$ are both contained in the kernel of $\bar{\rho}$. Since $\bar{\rho}(z)$ is periodic, so too are $\bar{\rho}(\alpha_1)$ and $\bar{\rho}(\alpha_2)$. By the classification of periodic elements in $\bar{B}_m$, each is conjugate to a power of either $\bar{\alpha}_1$ or $\bar{\alpha}_2$. Since $\bar{\alpha}_1$ has order $m$ and $\bar{\alpha}_2$ has order $m-1$, the claim follows.

We next claim that at least one of $\alpha_1^{m(m-1)}$ and $\alpha_2^{m(m-1)}$ is non-central. Since $n$ is coprime to $n-1$ and $n(n-1)$ does not divide $m(m-1)$, the Chinese remainder theorem implies that either $n$ does not divide $m(m-1)$ or $n-1$ does not divide $m(m-1)$. If $n$ does not divide $m(m-1)$, then $\alpha_1^{m(m-1)}$ is non-central in $B_n$. If $n-1$ does not divide $m(m-1)$, then $\alpha_2^{m(m-1)}$ is non-central in $B_n$. This completes the proof of the claim.

By the previous claim, there is a $k \in \{1,2\}$ so that $\alpha_k^{m(m-1)}$ is not central. Any such $\alpha_k^{m(m-1)}$ satisfies Lemma 4.2, and so its normal closure contains $B_n'$. By the first claim, this normal closure is contained in the kernel of $\bar{\rho}$, and so the image of $\bar{\rho}$ is cyclic, as desired.

We now proceed to the second statement. We first show that if $\bar{\rho}(\alpha_1)$ is equal to a power of $\bar{\alpha}_2$ then $\bar{\rho}$ has cyclic image. Then we show that if $\bar{\rho}(\alpha_1)$ is equal to $\bar{\alpha}_1^k$ with $\gcd(k,n) \neq 1$ then $\bar{\rho}$ has cyclic image. Up to conjugation in $B_m$, these are all possible cases, and the proposition follows from these two statements.

Assume that $\bar{\rho}(\alpha_1) = \bar{\alpha}_2^k$ for some integer $k$. Since $\alpha_1^n = \alpha_2^{n-1} = z$ it follows that $\bar{\rho}(\alpha_1)^n = \bar{\rho}(\alpha_2)^{n-1}$, and so $\bar{\alpha}_2^{kn} = \bar{\rho}(\alpha_1)^n = \bar{\rho}(\alpha_2)^{n-1}$. In the next paragraph we will use that fact that $\bar{\alpha}_2^{kn} = \bar{\rho}(\alpha_2)^{n-1}$.

Consider the composition

$$\bar{B}_n \xrightarrow{p} \mathbb{Z}/(n-1)n \to \mathbb{Z}/(n-1)$$

where $p$ denotes the abelianization map. Since the image of a half-twist under $p$ is 1 and since $\bar{\alpha}_2$ is a product of $n$ half-twists, it follows that $p(\bar{\alpha}_2) = n$. We now apply the given composition of homomorphisms to the equality $\bar{\alpha}_2^{kn} = \bar{\rho}(\alpha_2)^{n-1}$. The image of the left hand side is $kn^2$ and the image of the right-hand side is 0. In other words, $kn^2 \equiv k \equiv 0 \mod n-1$. Thus $\bar{\rho}(\alpha_1) = \bar{\alpha}_2^k = 1$. Applying Lemma 4.2, it follows that $\bar{\rho}$ has cyclic image, as desired.

Assume now that $\bar{\rho}(\alpha_1) = \alpha_1^k$ with $\gcd(k,n) \neq 1$. If $k = 0$ then $\alpha_1$ lies in $\ker \rho$. As above, it follows from Lemma 4.2 that $\bar{\rho}$ has cyclic image. So suppose now that $k > 0$. For any natural numbers $k$ and $n$ we have that $n$ divides $kt$, where $t = n/\gcd(n,k)$. In particular, for this choice of $t$ we have that $\alpha_1^t$ lies in $\ker(\bar{\rho})$. Since $\gcd(n,k) \neq 1$ we have that $t < n$, and so $\alpha_1^t$ is non-central. As in the case $m > n$ it follows that $\bar{\rho}$ has cyclic image, as desired. □

We have the following corollary of (the first statement of) Proposition 4.1.

**Corollary 4.3.** *Let $n \geq 5$, let $m \leq 2n$, and assume $m \neq n$. Let $\rho : B_n \to B_m$ be a homomorphism, and assume that $\bar{\rho}(z)$ is periodic. Then $\bar{\rho}$ has cyclic image.*

*Proof.* Under the assumptions on $m$ and $n$, it must be that $n(n-1)$ fails to divide $m(m-1)$. Indeed, suppose that $n(n-1)$ divides $m(m-1)$. In this case we must have $n \leq m$ and hence—since we are assuming $m \neq n$ and $m \leq 2n$—that $n < m \leq 2n$. Also, since $m$ and

$m-1$ are relatively prime, it must be that $n$ divides either $m$ or $m-1$. If $n$ divides $m$ then we must have $m = 2n$, and we see that $n(n-1)$ does not divide $m(m-1)$, a contradiction. If $n$ divides $m-1$, then—since $m \leq 2n$—the only possibility is that $m-1 = n$. Again in this case $n(n-1)$ does not divide $m(m-1)$, and the corollary follows by contradiction. □

## 5. Torsion to pseudo-Anosov

The goal of this section is to prove the following proposition. As above, we denote by $S_{0,n+1}$ a sphere with $n+1$ marked points and we identify the group $\bar{B}_n$ with the subgroup of $\mathrm{Mod}(S_{0,n+1})$ fixing one distinguished marked point $p$. Also, we say that an element of $\bar{B}_n$ is pseudo-Anosov if the corresponding element of $\mathrm{Mod}(S_{0,n+1})$ is.

**Proposition 5.1.** *Let $m, n \geq 3$, let $\rho : B_n \to B_m$ be a homomorphism. If $\bar{\rho}(z)$ is pseudo-Anosov then $\bar{\rho}$ has cyclic image.*

Before we can prove the proposition, we will need to establish the following lemma, which concerns the structure of the centralizer of a pseudo-Anosov braid.

**Lemma 5.2.** *Assume that $f \in \bar{B}_n$ is pseudo-Anosov. Then the centralizer of $f$ is abelian. If $g \in \bar{B}_n$ is also pseudo-Anosov and commutes with $f$, then the centralizer of $g$ equals the centralizer of $f$.*

*Proof.* In any group, two commuting elements with abelian centralizers have equal centralizers. Thus, the second statement follows from the first. So it remains to prove the first statement.

Suppose that $f \in \bar{B}_n$ is abelian. We may regard $f$ as an element of $\mathrm{Mod}(S_{0,n+1})$. Under the action of $\mathrm{Mod}(S_{0,n+1})$ on the space $\mathrm{PMF}(S_{0,n+1})$ of projective measured foliations, the element $f$ has two fixed points, $\mathcal{F}_s$ and $\mathcal{F}_u$. Let $\mathcal{G}$ denote the stabilizer in $\mathrm{Mod}(S_{0,n+1})$ of this (unordered) pair and let $\mathcal{G}^*$ denote the subgroup of $\mathcal{G}$ inducing the trivial permutation of $\{\mathcal{F}_s, \mathcal{F}_u\}$.

McCarthy proved there is a short exact sequence

$$1 \to F \to \mathcal{G}^* \to \mathbb{Z} \to 1$$

where $F$ is a finite group [10].

Any element $h$ of the centralizer of $f$ in $\mathrm{Mod}(S_{0,n+1})$ must fix the set of fixed points $\{\mathcal{F}_s, \mathcal{F}_u\}$; in other words, it must lie in $\mathcal{G}$. Further, since a pseudo-Anosov mapping class acts with source-sink dynamics on $\mathrm{PMF}(S_{0,n+1})$, it follows that $h$ must preserve the two fixed points (the source and the sink)) individually; in other words it must lie in $\mathcal{G}^*$. Since the centralizer of $f$ in $\bar{B}_n$ is contained in the centralizer of $f$ in $\mathrm{Mod}(S_{0,n+1})$, it suffices to show that $\mathcal{G}^* \cap \bar{B}_n$ is abelian.

To this end, we restrict the above short exact sequence to $\bar{B}_n$:

$$1 \to F_0 \to \mathcal{G}_0^* \to \mathbb{Z} \to 1.$$

(Since the homomorphism $\mathcal{G}^* \to \mathbb{Z}$ records the stretch factor the restricted homomorphism is still surjective, as indicated; however, we will not use this.) The group $F_0$—indeed any finite subgroup of $\bar{B}_n$ may be regarded as a group of rotations about $p$. It follows that, up to conjugacy, $F_0$ is a subgroup of either $\langle \bar{\alpha}_1 \rangle$ or $\langle \bar{\alpha}_2 \rangle$. For a given $k$, no two distinct elements of $\langle \bar{\alpha}_k \rangle$ are conjugate in $\bar{B}_n$; this can be seen, for example, from the abelianization of $\bar{B}_n$. In particular, no two distinct elements of $F_0$ are conjugate in $\bar{B}_n$.

The group $\mathcal{G}_0^*$ acts on the cyclic group $F_0$ by conjugation. Since no two distinct elements of $F_0$ are conjugate in $\bar{B}_n$ (they have distinct images in the abelianization of $\bar{B}_n$), the action is trivial. It follows that $F_0$ is central in $\mathcal{G}_0^*$ and hence that $\mathcal{G}_0^*$ is abelian, as desired. □

We are now ready to prove Proposition 5.1.

*Proof of Proposition 5.1.* The group $B_n$ is equal to the centralizer of $z$ and hence $\bar{\rho}(B_n)$ maps into the centralizer of the pseudo-Anosov element $\bar{\rho}(z)$. By Lemma 5.2, the latter is abelian. Since the abelianization of $\bar{B}_n$ is cyclic, the proposition follows. □

## 6. The interior/exterior decomposition

In this section we introduce a basic tool for handling cablings $\rho : B_n \to B_m$. As in the introduction, the cabling maps are exactly the ones that are reducible in the sense that there is some multicurve $M$ preserved by $\rho(B_n)$. We first address the case where each component of $M$ is fixed, and then the more general case.

6.1. **Braids fixing a multicurve.** Let $M = \{c_1, \ldots, c_k\}$ be an un-nested multicurve in $\mathbb{D}_m$. Say that each $c_i$ surrounds exactly $p_i$ marked points, and let $p = p_1 + \cdots + p_k$. Let $\mathrm{Fix}_{B_m}(M)$ denote the subgroup of $B_m$ given by the intersection of the stabilizers of the $c_i$.

We consider two homomorphisms:

$$\Pi_i : \mathrm{Fix}_{B_m}(M) \to B_{p_1} \times \cdots \times B_{p_k}, \text{ and}$$
$$\Pi_e : \mathrm{Fix}_{B_m}(M) \to B_{m-p+k}.$$

The $i$th component of the homomorphism $\Pi_i$ is the one obtained by forgetting the marked points in the exterior of $c_i$, and the homomorphism $\Pi_e$ is the one given by collapsing the disks bounded by the $c_i$ to marked points.

For any $n$ we denote by $B_{n,n'}$ the subgroup of $B_{n+n'}$ consisting of braids that fix the the first $n'$ marked points. The image of $\Pi_e$ lies in the subgroup of $B_{m-p+k}$ consisting of the elements that fix the marked points corresponding to the $c_i$. We identify this subgroup with $B_{m-p,k}$ .

**Lemma 6.1.** *Let $M = \{c_1, \ldots, c_k\}$ be an un-nested multicurve in $\mathbb{D}_m$. Say that each $c_i$ surrounds exactly $p_i$ marked points, and let $p = p_1 + \cdots + p_k$. The homomorphism*

$$\Pi_i \times \Pi_e : \mathrm{Fix}_{B_m}(M) \to (B_{p_1} \times \cdots \times B_{p_k}) \times B_{m-p,k}$$

*is an isomorphism.*

*Proof.* We would like to define an inverse homomorphism to $\Pi_i \times \Pi_e$. The key point is that there is a homomorphism $\beta : B_{m-p,k} \to \mathrm{Stab}_{B_m}(M)$ with $\Pi_e \circ \beta$ is equal to the identity; geometrically we can think of $\beta$ as the map defined by "blowing up" each of the the first $k$ marked points to a disk with $p_i$ marked points (alternatively, in the braid picture, each of the first $k$ strands is replaced with a cable in such a way that there are no crossings of strands inside the cables). Bell and the third author [7] constructed such a homomorphism (this is essentially the map $\iota : L_k \to \overline{\mathrm{Mod}}(\bar{S}_k)$ from Figure 7 in their paper; in order to obtain $\beta$ from their map we should post-compose $\iota$ with the map $\overline{\mathrm{Mod}}(\bar{S}_k) \to B_m$ induced by inclusion of $\bar{S}_k$ into $\mathbb{D}_m$).

For $1 \le j \le k$, let $\iota_j : B_{p_j} \to B_m$ be the standard inclusion with image supported in the interior of $c_j$. The image of $\iota_j$ lies in $\mathrm{Fix}_{B_m}(M)$.

We define

$$F : (B_{p_1} \times \cdots \times B_{p_k}) \times B_{m-k,p} \to \mathrm{Fix}_{B_m}(M)$$

by

$$F(f_1, \ldots, f_k, f_e) = \iota_1(f_1)\iota_2(f_2)\cdots\iota_k(f_k)\beta(f_e).$$

The map $F$ is a homomorphism because the images of the maps $\iota_1, \ldots, \iota_k, \beta$ commute pairwise. From the way the maps are defined, it is clear that $(\Pi_i \times \Pi_e) \circ F$ is the identity. □

Given a homomorphism $\rho : B_n \to B_m$ and an un-nested multicurve $M$ fixed by $\rho(B_n)$ as above, we define

$$\rho_i : B_n \to B_{p_1} \times \cdots \times B_{p_k}, \text{ and}$$
$$\rho_e : B_n \to B_{m-p,k}$$

by the formulas $\rho_i = \Pi_i \circ \rho$ and $\rho_e = \Pi_e \circ \rho$. We refer to these as the *interior* and *exterior* components of $\rho$.

6.2. **Braids preserving a multicurve.** Let $M = \{c_1, \dots, c_k\}$ be a multicurve in $\mathbb{D}_m$. Suppose that each $c_i$ surrounds exactly $p$ marked points, and suppose that $kp = m$. These conditions imply that every marked point in $\mathbb{D}_m$ lies in the interior of exactly one $c_i$ and in particular that the $c_i$ are un-nested. Let $\mathrm{Stab}_{B_m}(M)$ denote the subgroup of $B_m$ consising of elements that preserve $M$. We will give a semi-direct product decomposition of $\mathrm{Stab}_{B_m}(M)$ that is analogous to the direct product decomposition for $\mathrm{Fix}_{B_m}(M)$ given in Lemma 6.1.

Consider the homomorphism

$$\Pi_e : \mathrm{Stab}_{B_m}(M) \to B_k,$$

given by collapsing the disks bounded by the $c_i$ to marked points. It follows from the assumptions on the $c_i$ that $\Pi_e$ is surjective. The kernel of $\Pi_e$ is the product of the braid groups corresponding to the interiors of the $c_i$ (this follows from [18, Theorem 3.18]). We thus have a short exact sequence

$$1 \to \prod_{i=1}^{k} B_p \to \mathrm{Stab}_{B_m}(M) \to B_k \to 1.$$

There is a splitting $B_k \to \mathrm{Stab}_{B_m}(M)$ given by a variant of the homomorphism $\beta$ from the proof of Lemma 6.1. This variant is also derived from the map $\iota$ defined by Bell and the third author. The action of $B_k$ on $\prod B_p$ factors through the standard map $B_k \to S_k$ and is given by permutation of the factors. We thus have the following lemma.

**Lemma 6.2.** *Let $M = \{c_1, \dots, c_k\}$ be an un-nested multicurve in $\mathbb{D}_m$. Say that each $c_i$ surrounds exactly $p$ marked points, and suppose $m = kp$. There is a split short exact sequence*

$$1 \to \prod_{i=1}^{k} B_p \to \mathrm{Stab}_{B_m}(M) \xrightarrow{\Pi_e} B_k \to 1.$$

*In particular we have an isomorphism*

$$\mathrm{Stab}_{B_m}(M) \cong B_k \ltimes \prod_{i=1}^{k} B_p,$$

*where $B_k$ acts by permuting the factors of $\prod B_p$ according to the standard map $B_k \to S_k$.*

Lemma 6.1 is a consequence of Lemma 6.2. It is possible to further generalize Lemma 6.2 to the case where $M$ is an arbitrary un-nested multicurve, but we will not need this more general statement.

## 7. Castel's theorem and extensions

In this section we give our proof of Castel's theorem, which classifies homomorphisms $B_n \to B_n$ for $n \geq 6$. In addition to extending Castel's result to the case $n \geq 5$ (Theorem 7.1), we prove as a separate theorem an extension to the case $n = 4$, which accounts for the exceptional homomorphism $B_4 \to B_3$ (Theorem 7.5). We prove each of these theorems in a separate subsection.

### 7.1. Castel's theorem.

The following theorem is our extension of Castel's theorem to the case $n \geq 5$. At the end of the section we give Corollary 7.4, a repackaging of Theorem 7.1 that will be used in the proof of Theorem 1.1.

**Theorem 7.1.** *Let $n \geq 5$. Any homomorphism $\rho : B_n \to B_n$ is equivalent to either the trivial homomorphism or the identity.*

Before proceeding to the proof of Theorem 7.1, we require two preliminary lemmas.

**Lemma 7.2.** *Let $m, n \geq 3$, and let $\rho : B_n \to B_m$ be a homomorphism. Then $\rho$ has cyclic image if and only if $\bar{\rho}$ has cyclic image.*

*Proof.* If $\rho$ has cyclic image then $\bar{\rho}$ must be cyclic, since $\bar{\rho}$ is the post-composition of $\rho$ with a quotient map. Now assume that $\bar{\rho}$ has cyclic image. Then we have an exact sequence

$$1 \to Z_m \cap \rho(B_n) \to \rho(B_n) \to \bar{\rho}(B_n) \to 1,$$

so $\rho(B_n)$ is an extension of a cyclic group by a cyclic group. Since $Z_m \cap \rho(B_n)$ is central in $B_m$, we in fact have that $\rho(B_n)$ is abelian. This implies that $\rho$ factors through the abelianization of $B_n$, hence it has cyclic image. □

**Lemma 7.3.** *Let $n \geq 5$ and $m \geq 3$, and let $\rho : B_n \to B_m$ a homomorphism. If the canonical reduction system of $\bar{\rho}(\sigma_1)$ is empty, then $\bar{\rho}$ has cyclic image.*

*Proof.* The hypothesis implies that $\bar{\rho}(\sigma_1)$ is either periodic or pseudo-Anosov. We treat these two cases in turn.

Assume that $\bar{\rho}(\sigma_1)$ is periodic. Since $\bar{\rho}(\sigma_3)$ commutes with $\bar{\rho}(\sigma_1)$, together they generate a finite abelian subgroup of $\bar{B}_m$. As in the proof of Lemma 5.2, any such subgroup is conjugate to a subgroup of either $\langle \bar{\alpha}_1 \rangle$ or $\langle \bar{\alpha}_2 \rangle$. Since no two distinct elements of $\langle \bar{\alpha}_1 \rangle \cup \langle \bar{\alpha}_2 \rangle$ are conjugate it follows that $\bar{\rho}(\sigma_1) = \bar{\rho}(\sigma_3)$. Since $n \geq 5$, the commutator subgroup $B_n'$ is normally generated in $B_n$ by $\sigma_1 \sigma_3^{-1}$, and so it follows that $\bar{\rho}$ has cyclic image.

Now assume that $\bar{\rho}(\sigma_1)$ is pseudo-Anosov. Since each $\bar{\rho}(\sigma_i)$ is conjugate to $\bar{\rho}(\sigma_1)$, the $\bar{\rho}(\sigma_i)$ are all pseudo-Anosov. Since each $\bar{\rho}(\sigma_{2i-1})$ commutes with $\bar{\rho}(\sigma_1)$, they all have the same centralizer by Lemma 5.2. Since $n \geq 5$, each $\bar{\rho}(\sigma_{2j})$ commutes with some $\bar{\rho}(\sigma_{2i-1})$; hence all of the $\bar{\rho}(\sigma_i)$ have the same centralizer. In particular, they all commute with each other. It follows that the image of $\bar{\rho}$ is abelian. Since the abelianization of $B_n$ is cyclic, it follows that $\bar{\rho}$ has cyclic image. □

*Proof of Theorem 7.1.* We consider three cases, according to whether $\bar{\rho}(z)$...

(1) is pseudo-Anosov,
(2) is periodic, or
(3) has non-empty canonical reduction system.

We treat the three cases in turn.

*Case 1: $\bar{\rho}(\alpha_1)$ is pseudo-Anosov.* By Proposition 5.1, we have that $\bar{\rho}$ has cyclic image. By Lemma 7.2, $\rho$ itself has cyclic image.

*Case 2: $\bar{\rho}(\alpha_1)$ is periodic.* Assume that $\bar{\rho}$ is not cyclic. We will show that $\rho$ is equivalent to the identity map. By Proposition 4.1(2), we may assume that $\bar{\rho}(\alpha_1)$ is equal to $\bar{\alpha}_1^k$, where $\gcd(k,n) = 1$.

By Lemma 7.3, the canonical reduction system $M$ of $\bar{\rho}(\sigma_1)$ is non-empty. The key to the proof in this case is to show that $M$ is a single curve surrounding exactly two marked points. To do this, we will apply Proposition 3.3.

We claim that $k \in \{\pm 1\}$. Suppose not. Since $\gcd(k,n) = 1$, it follows that $k$ is a unit in $\mathbb{Z}/n$, and so it has an inverse $j$. The inverse $j$ does not lie in $\{\pm 1\}$, and so $\bar{\alpha}_1^j \sigma_1 \bar{\alpha}_1^{-j}$ commutes with $\sigma_1$. Applying $\bar{\rho}$ we conclude that $\bar{\alpha}_1 \bar{\rho}(\sigma) \bar{\alpha}_1^{-1}$ commutes with $\bar{\rho}(\sigma_1)$. It follows that $i(\bar{\alpha}_1(M), M) = 0$. This violates Proposition 3.1 and so the claim is proven.

By the previous claim, we may assume that $\bar{\rho}(\alpha_1) = \bar{\alpha}_1$ (if necessary, we post-compose with the inversion automorphism of $B_n$).

Next we claim that $M$ is a single curve surrounding exactly two marked points. Since $\alpha_1^t \sigma_1 \alpha_1^{-t}$ commutes with $\sigma_1$ for $t \in \{2, \ldots, n-2\}$, it follows that $\bar{\rho}(\alpha_1^t)\bar{\rho}(\sigma_1)\bar{\rho}(\alpha_1)^{-t}$ commutes with $\bar{\rho}(\sigma_1)$ for $t \in \{2, \ldots, n-2\}$. By the previous claim, it further follows that $\bar{\alpha}_1^t \bar{\rho}(\sigma_1) \bar{\alpha}_1^{-t}$ commutes with $\bar{\rho}(\sigma_1)$ for $t \in \{2, \ldots, n-2\}$. This means that $i(\bar{\alpha}_1^t(M), M) = 0$ for $t \in \{2, \ldots, n-2\}$. The claim now follows from Proposition 3.3.

Let $c_i$ denote the canonical reduction system of $\bar{\rho}(\sigma_i)$ for $1 \le i \le n-1$. By the previous paragraph, $c_1 = M$ is a single curve surrounding exactly two marked points. Since the $\sigma_i$ are all conjugate in $B_n$, each $c_i$ is a single curve surrounding exactly two marked points. Since $\sigma_1$ commutes with $\sigma_i$ for $i \ge 3$, it follows that $c_1$ and $c_i$ have trivial intersection for each such $i$. Since they surround the same number of marked points, they are not nested.

We claim that $\bar{\rho}(\sigma_1)$ is a power of the half-twist about $c_1$. Since $c_1$ is the canonical reduction system for $\bar{\rho}(\sigma_1)$, the mapping class $\bar{\rho}(\sigma_1)$ acts as either a periodic or a pseudo-Anosov mapping class on the exterior region $R$ of $c_1$. The claim is equivalent to the statement that this mapping class is trivial. But this follows from the fact that $\bar{\rho}(\sigma_1)$ fixes the curve $c_3$ and the fact that $c_1$ and $c_3$ are not nested.

We claim that the $c_i$ are distinct. If not, then since the $\sigma_i$ are all conjugate there are distinct $i$ and $j$ with $\bar{\rho}(\sigma_i) = \bar{\rho}(\sigma_j)$. It follows that $\sigma_i \sigma_j^{-1}$ lies in the kernel of $\bar{\rho}$. Since the normal closure of any $\sigma_i \sigma_j^{-1}$ is $B_n'$ whenever $i \ne j$, it then follows then $\bar{\rho}$, hence $\rho$, has cyclic image (Lemma 7.2). This is a contradiction.

Since $\bar{\rho}(\sigma_1) = H_{c_1}^\ell$, we have that $\rho(\sigma_i) = H_{c_i}^\ell z^m$ for all $i$. Since $\sigma_i$ and $\sigma_{i+1}$ satisfy the braid relation, so too must $H_{c_i}^\ell$ and $H_{c_{i+1}}^\ell$. Bell and the third author proved [6, Lemma 4.9] that if $H_{c_i}^\ell$ and $H_{c_{i+1}}^\ell$ satisfy the braid relation and $c_i \ne c_{i+1}$ then $i(c_i, c_{i+1}) = 2$ and $\ell = \pm 1$. Thus up to an automorphism of $B_m$ we have $\rho(\sigma_i) = \sigma_i z^m$ (apply the previous claim). Then, up to a transvection, we have $\rho(\sigma_i) = \sigma_i$. This completes the proof of the second case.

*Case 3: $\bar{\rho}(\alpha_1)$ has non-empty canonical reduction system.* For this case we work directly with $\rho$ instead of $\bar{\rho}$. Since $\bar{\rho}(\alpha_1)$ has non-empty canonical reduction system, so does $\rho(\alpha_1)$. We denote the canonical reduction system of $\rho(\alpha_1)$ by $M$. Since $z$ is a power of $\alpha_1$, it follows that $M$ is also equal to the canonical reduction system of $\rho(z)$.

Since $\rho(B_n)$ lies in the centralizer of $\rho(z)$, it follows that, through $\rho$, the group $B_n$ acts by permutations on the set of components of $M$.

We claim that the action of $B_n'$ on the set of components of $M$ is trivial. Since the maximum number of components of a multicurve in $\mathbb{D}_n$ is $n-2$, the number of components of $M$ is at most $n-2$. Lin proved [32, Theorem A] that any homomorphism $B_n \to S_k$ with $k < n$ has cyclic image if $n \geq 5$; thus the action of $B_n$ on the set of components of $M$ is cyclic. The claim follows.

Let $c$ be a component of $M$, and say that $c$ surrounds exactly $p$ marked points. We will use the notation $\mathrm{Fix}_{B_n}(c)$, $\Pi_i$, and $\Pi_e$ from Section 6. By the previous claim, $\rho(B_n')$ lies in the group $\mathrm{Fix}_{B_n}(c)$. Lin proved that for $n \geq 5$ any homomorphism $B_n' \to B_k$ with $k < n$ is trivial when $n \geq 5$ [32, Theorem A]. It follows that the groups $\Pi_i \circ \rho(B_n')$ and $\Pi_e \circ \rho(B_n')$ are trivial, as they are the images of homomorphisms that satisfy the hypotheses of Lin's theorem. Since the map $\Pi_i \times \Pi_e$ is injective (Lemma 6.1), it follows that $\rho(B_n')$ is trivial, and so $\rho$ has cyclic image, as desired. □

We have the following corollary of Theorem 7.1 (the theorem also follows quickly from the corollary, and so we may think of the corollary as a slightly different version of the theorem). We will apply this corollary in our proof of Theorem 1.1. In the statement, the inversion map $\iota : B_n \to B_n$ is the map given by $\iota(\sigma_i) = \sigma_i^{-1}$ for all $i$. This induces the nontrivial outer automorphism of $B_n$.

**Corollary 7.4.** *Let $n \geq 5$ and let $\rho : B_n \to B_n$ be a homomorphism. Then there is an inner automorphism $\tau$ of $B_n$, and a $t \in B_n$ so that $\tau \circ \rho^t$ is equal to the trivial map, the inversion map, or the identity.*

*Proof.* If $\rho$ has cyclic image, then there is an $f \in B_n$ so that $\rho(\sigma_i) = f$ for all $i$. Then the conclusion of the theorem holds with $\tau$ equal to the identity and with $t = f^{-1}$, as the map $\tau \circ \rho^t$ is then the trivial map.

Now assume that $\rho$ does not have cyclic image. By Theorem 7.1 and the definition of the equivalence relation on homomorphisms, there is an automorphism $\tau_0$ and a $t_0$ in $B_n$ so that $\rho = \tau_0 \circ id^{t_0}$. Since $\mathrm{Out}(B_n) \cong \mathbb{Z}/2$, we may write $\tau$ as $\tau_1 \circ \iota$. where $\tau_1$ is an inner automorphism. So we have

$$\rho = \tau_1 \circ \iota \circ id^{t_0} = \tau_1 \iota^{\iota(t_0)},$$

as desired. □

7.2. **The case of four strands.** In this section we state and prove the following analogue of Castel's theorem for the case of $B_4$. A new proof of this theorem was recently given by Orevkov [36, Theorem 1.7]

**Theorem 7.5.** *Let $\rho : B_4 \to B_4$ be a homomorphism. Then either*

- *$\rho$ factors through the standard homomorphism $B_4 \to B_3$ or*
- *$\rho$ is equivalent to either the trivial map or the identity map.*

The proof follows the same lines as our proof of Theorem 7.1. There are three ways in which the argument differs. The first issue is that the normal closure of $\sigma_1\sigma_3^{-1}$ in $B_4$ is not the commutator subgroup, but rather the kernel of the standard homomorphism $B_4 \to B_3$. Since we allow for this homomorphism in the statement of Theorem 7.5, this means that we can use the same arguments as in the proof of Theorem 7.1, just with a different conclusion. Specifically, this issue arises in Lemma 7.3 and in Lemma 4.2 (the latter is used in the proof of Proposition 4.1).

The second issue is that Proposition 3.3 does not hold as stated for $n = 4$. We will require a specialized version for $n = 4$. There is one additional possibility, namely, that $M$ is a multicurve with two components, each surrounding two marked point. The proof of this version is essentially the same as for the $n \geq 5$ case.

The third issue is that, in the case where we have a cabling $B_4 \to B_3$, we cannot conclude that the image is cyclic, again because of the standard homomorphism $B_4 \to B_3$. To deal with this issue, we must first classify homomorphisms $B_4 \to B_3$. We begin with this classification, and then use it to prove Theorem 7.5.

Before proceeding to the classifications of homomorphisms $B_4 \to B_3$ we first prove the following lemma, which is the analogue of Theorem 7.2 for the case $n = 4$.

**Lemma 7.6.** *Let $m \geq 1$ and let $\rho : B_4 \to B_m$ be a homomorphism. Suppose that some $\sigma_i\sigma_j^{-1}$ lies in the kernel of $\bar{\rho}$. Then either $\rho$ is cyclic or it factors through the standard map $B_4 \to B_3$.*

*Proof.* Suppose first that $\sigma_1\sigma_2^{-1}$ or $\sigma_2\sigma_3^{-1}$ lies in the kernel of $\bar{\rho}$. The normal closure of either contains $B_4'$. It follows that the image of $\bar{\rho}$ is cyclic. As in the proof of Lemma 7.2, the image of $\rho$ is cyclic.

If $\sigma_1\sigma_3^{-1}$ lies in the kernel of $\bar{\rho}$, then $\rho(\sigma_1\sigma_3^{-1})$ is central in $B_m$. Since $\rho(B_4')$ is contained in $B_m'$ (the image of a product of commutators is a product of commutators) and since $Z(B_m) \cap B_m'$ is trivial, it follows that $\sigma_1\sigma_3^{-1}$ lies in the kernel of $\rho$. Thus, $\rho$ factors through the standard homomorphism $B_4 \to B_3$. □

**Proposition 7.7.** *Let $\rho : B_4 \to B_3$ be a homomorphism. Then $\rho$ is equivalent to either the trivial map or it factors through the standard map $B_4 \to B_3$. Further, if $\rho$ is the composition of the standard homomorphism with an automorphism of $B_3$.*

*Proof.* The proof follows the same outline as the proof of Theorem 7.1. We consider three cases for $\bar{\rho}(z)$, according to whether it is pseudo-Anosov, periodic, or reducible.

If $\bar{\rho}(z)$ is pseudo-Anosov then it follows from Proposition 5.1 that $\rho$ has cyclic image.

If $\bar{\rho}(z)$, hence $\bar{\rho}(\alpha_1)$, is periodic, then it follows that $\bar{\rho}(\alpha_1)$ has order 1, 2, or 3. We treat these three possibilities in turn. By Lemma 7.6, it suffices to show in each case that some $\sigma_i\sigma_j^{-1}$ lies in the kernel of $\bar{\rho}$.

If $\bar{\rho}(\alpha_1)$ has order 1, this means that $\alpha_1$ lies in the kernel of $\bar{\rho}$, and (as in the proof of Proposition 4.1(2)), the kernel of $\bar{\rho}$ contains $\sigma_1\sigma_2^{-1}$, as desired. If $\bar{\rho}(\alpha_1)$ has order 2, $\alpha_1^2$ lies in the kernel of $\bar{\rho}$, and it follows that the kernel of $\bar{\rho}$ contains $\sigma_1\sigma_3^{-1}$, as desired. If $\bar{\rho}(\alpha_1)$ has order 3, we have that $\alpha_1^3$ lies in the kernel of $\bar{\rho}$. Thus there is a conjugate of $\sigma_1\sigma_2^{-1}$ in the kernel of $\bar{\rho}$, again as desired.

Finally, suppose that $\bar{\rho}(z)$, hence $\rho(z)$, is reducible. In this case, there is a multicurve $M$ preserved by $\rho(B_4)$. Since $M$ lies in $\mathbb{D}_3$, it must be that $M$ has a single component with exactly two marked points in the interior. Further, $\rho(B_4)$ lies in $\mathrm{Fix}_{B_3}(M)$, which by Lemma 6.1 is isomorphic to $B_{1,1} \times B_2 \cong \mathbb{Z} \times \mathbb{Z}$. In particular, the image of $\rho$ is abelian, hence cyclic. This completes the proof of the first statement. The second statement follows from the first statement and the fact that $B_n$ enjoys the Hopfian property for all $n$; in other words, every surjective homomorphism is an isomorphism. □

We are now ready for the proof of Theorem 7.5. The most difficult part of the proof is the case where $\rho$ is a cabling and the reducing multicurve has two components (this is what comes from the extension of Proposition 3.3 to the case $n = 4$). This case parallels Case 3 of the proof of Theorem 1.1, which is the most difficult part of that proof. To deal with this case, we study the Garside element $\Delta \in B_4$, which is conjugate to—but not equal to—$\alpha_1^2$. The interplay between these periodic elements leads to a contradiction.

*Proof of Theorem 7.5.* Again, there are three possibilities for $\bar{\rho}(z)$: it can be pseudo-Anosov, periodic, or reducible.

In the pseudo-Anosov case, the argument is exactly the same as in the proof of Theorem 7.1. In the reducible case, the argument is essentially the same, except we must apply

Proposition 7.7 in the case where the reducing multicurve has exactly one component, which has exactly three marked point in the interior.

It remains to consider the case where $\rho(z)$, hence $\rho(\alpha_1)$, is periodic. We may assume that $\rho$ does not have cyclic image and that it does not factor through the standard homomorphism $B_4 \to B_3$. By Lemma 7.6, this is equivalent to the assumption that no $\sigma_i\sigma_j^{-1}$ lies in the kernel of $\bar{\rho}$.

The argument of Proposition 4.1(2) shows that either $\sigma_1\sigma_2^{-1}$ lies in the kernel of $\bar{\rho}$, or $\sigma_1\sigma_3^{-1}$ lies in the kernel of $\bar{\rho}$, or $\bar{\rho}(\alpha_1)$ is equal to $\bar{\alpha}_1^k$, where $\gcd(k,n) = 1$.

By Lemma 7.6, we may assume that $\bar{\rho}(\alpha_1) = \bar{\alpha}_1^k$ with $\gcd(k,4) = 1$. In particular $k = \pm 1$ mod 4. We may further assume (after possibly post-composing with the inversion automorphism of $B_4$) that $k = 1$, and so $\rho(\alpha_1) = \alpha_1$.

We now claim that $\bar{\rho}(\sigma_1)$ has nonempty canonical reduction system $M$. By the same argument as in the proof of Lemma 7.3, we see that if $\bar{\rho}(\sigma_1)$ is periodic then $\sigma_1\sigma_3^{-1}$ lies in the kernel of $\rho$, a contradiction. For the case where $\bar{\rho}(\sigma_1)$ is pseudo-Anosov, we use a variant of the argument used to prove Lemma 7.3 (this variant also applies in the case $n \geq 5$). Since $\bar{\rho}(\sigma_1)$ and $\bar{\rho}(\sigma_3)$ are commuting, conjugate pseudo-Anosov mapping classes, they have the same invariant foliations and stretch factors. It follows that $\bar{\rho}(\sigma_1\sigma_3^{-1})$, hence $\rho(\sigma_1\sigma_3^{-1})$ is periodic. But since $\sigma_1\sigma_3^{-1}$ lies in $B_4'$, it must be that $\rho(\sigma_1\sigma_3^{-1})$ lies in $B_4'$. The only periodic element of $B_n'$ is the identity. Again, this is a contradiction.

By applying a version of Proposition 3.3 for the case $k = 1$ and $n = 4$, the multicurve $M$ is either a single curve surrounding two marked points, or a pair of curves that each surround two marked points. Moreover, in the second case, it must be that the two curves are $c_1$ and $c_3$ or $c_0$ and $c_2$, where $c_1$, $c_2$, and $c_3$ are the standard curves with $H_{c_i} = \sigma_i$ and $c_0$ is the unique curve so that $c_0$, $c_1$, $c_2$, and $c_4$ form a square preserved by $\alpha_1$ (any other curve passes though this square in an essential way, and hence intersects its image under $\alpha_1^2$). Up to conjugating by $\alpha_1$ so that $M = \{c_1, c_3\}$ (this does not change the fact that $\rho(\alpha_1) = \alpha_1$). In the first case, the proof of the theorem concludes in the same way as the proof of Theorem 7.1. So we may assume that we are in the second case.

After post-composing $\rho$ with an inner automorphism, we may assume that the canonical reduction system of $\rho(\sigma_1)$ consists of the curves $c_1$ and $c_3$. There are two cases to consider, distinguished by whether or not $\rho(\sigma_1)$ identically fixes each of $c_1$ and $c_3$. We now complete the proof in two separate cases:

(1) the components of $M$ are both fixed by the action of $\rho(B_4)$, and
(2) the components of $M$ are interchanged by the action of $\rho(B_4)$.

We will argue by contradiction that neither of these cases occur.

For cases, our argument makes use of two specific braids, $\Delta$ and $\sigma_0$. The braid $\Delta$ is the Garside element $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$, and the braid $\sigma_0$ is $\sigma_0 = \alpha_1\sigma_3\alpha_1^{-1}$. The braid $\sigma_0$ is the half-twist about the curve $c_0$ defined above. In each of the two cases, we will derive a contradiction by showing that $\rho(\Delta)$ both commutes with $\sigma_0$ and does not commute with $\sigma_0$.

The braids $\Delta$ and $\sigma_0$ fail to commute (this claim will be used for both of the cases). Indeed, under the usual identification of $B_4$ with $\text{Mod}(\mathbb{D}_4)$ (where the marked points in $\mathbb{D}_4$ lie on the $x$-axis), the braid $\sigma_0$ is a half-twist about the arc $\gamma$ in the upper half-disk that connects the first and fourth marked points, and $\Delta$ is given by a rigid rotation by $\pi$ of a proper sub-disk of $\mathbb{D}_4$. It follows that $\Delta$ and $\sigma_0$ fail to commute.

We now deal with the first case. We begin with some setup. It follows from the description of $M$ and the assumption that $\rho(B_4)$ fixes each component of $M$ that $\rho(\sigma_1) = \sigma_1^a\sigma_3^bz^c$ for some integers $a$, $b$, and $c$. After modifying $\rho$ by the transvection determined by $z^{-c}$, we may assume that $c = 0$.

We claim that $a \neq b$. Since $\rho(\alpha_1) = \alpha_1$, we also have $\rho(\alpha_1^2) = \alpha_1^2$. We then conclude that

$$\rho(\sigma_3) = \rho(\alpha_1^2 \sigma_1 \alpha_1^{-2}) = \rho(\alpha_1^2)\rho(\sigma_1)\rho(\alpha_1^{-2}) = \alpha_1^2 \rho(\sigma_1)\alpha_1^{-2} = \alpha_1^2 \sigma_1^a \sigma_3^b \alpha_1^{-2} = \sigma_1^b \sigma_3^a.$$

Since we assumed that $\sigma_1\sigma_3^{-1}$ does not lie in the kernel of $\rho$, the claim follows.

We now claim that conjugation by $\rho(\Delta)$ interchanges $\sigma_1$ and $\sigma_3$. Since conjugation by $\Delta$ interchanges $\sigma_1$ and $\sigma_3$, conjugation by $\rho(\Delta)$ interchanges $\sigma_1^a\sigma_3^b$ and $\sigma_1^b\sigma_3^a$. Since $a \neq b$, the claim follows.

We next claim that $\rho(\Delta)$ commutes with both $\sigma_0$ and $\sigma_2$. Since $\sigma_2 = \alpha_1\sigma_1\alpha_1^{-1}$ and $\rho(\alpha_1) = \alpha_1$, we have that

$$\rho(\sigma_2) = \rho(\alpha_1\sigma_1\alpha_1^{-1}) = \rho(\alpha_1)\rho(\sigma_1)\rho(\alpha_1^{-1}) = \alpha_1(\sigma_1^a\sigma_3^b)\alpha_1^{-1} = \sigma_2^a\sigma_0^b.$$

Since $\Delta$ commutes with $\sigma_2$, we have that $\rho(\Delta)$ commutes with $\sigma_2^a\sigma_0^b$. Since $a \neq b$, we have that $\rho(\Delta)$ commutes with each of $\sigma_2$ and $\sigma_0$. In particular it commutes with $\sigma_0$, as per the claim.

We now claim that $\rho(\Delta) \equiv \Delta \mod Z(B_4)$. We already showed that $\rho(\Delta)$ conjugates $\sigma_1$ to $\sigma_3$ and vice versa. By the previous claim, $\rho(\Delta)$ conjugates $\sigma_2$ to itself. Since $\sigma_1$, $\sigma_2$, and $\sigma_3$ generate $B_4$, the claim follows.

Since $\rho(\Delta) = \Delta z^k$ for some $k$, it follows that $\rho(\Delta)$ does not commute with $\sigma_0$ (the braid $\Delta$ does not preserve $c_0$). This is the desired contradiction.

We now proceed to the second case. The assumptions of the second case imply that $\rho(\sigma_1^2)$ fixes each of $c_1$ and $c_3$. As above, there are integers $a$, $b$, and $c$ such that $\rho(\sigma_1^2) = \sigma_1^a\sigma_3^b z^c$. As in the first case, this implies that $\rho(\sigma_3^2) = \sigma_1^b\sigma_3^a z^c$ and $\rho(\sigma_2^2) = \sigma_2^a\sigma_0^b z^c$. Since $B_4$ is torsion free, the fact that $\sigma_1\sigma_3^{-1} \notin \ker\rho$ implies that $(\sigma_1^2\sigma_3^{-2}) = (\sigma_1\sigma_3^{-1})^2 \notin \ker\rho$. It again follows that $a \neq b$, and hence that $\rho(\Delta)$ induces an inner automorphism of $B_4$ that interchanges $\sigma_1$ and $\sigma_3$ and fixes both $\sigma_2$ and $\sigma_0$. Also as above, this inner automorphism cannot fix $\sigma_0$. This contradiction completes the proof of the theorem. □

## 8. The group of even braids

Recall that $B_n^2$ is the group of even braids. The goal of this section is to prove the following theorem, which may be viewed as an extension of Theorem 7.1.

**Theorem 8.1.** *Let $n \geq 5$, and let $\rho : B_n^2 \to B_n$ be a homomorphism. Then $\rho$ is equivalent to either the trivial homomorphism or to the standard inclusion map.*

As for the case of the full braid group we denote by $\bar{\rho}$ the homomorphism $\bar{\rho} : B_n^2 \to \bar{B}_n$ associated to a homomorphism $\rho : B_n^2 \to B_n$.

Before proceeding to the proof of Theorem 8.1 we require a series of lemmas.

**Lemma 8.2.** *Let $n \geq 2$. The commutator subgroup of $B_n^2$ is $B_n'$ and the abelianization $B_n^2/B_n'$ is isomorphic to $\mathbb{Z}$.*

*Proof.* Since $B_n'$ is contained in $B_n^2$, and since $B_n^2/B_n' \subset B_n/B_n' \cong \mathbb{Z}$ is equal to the subgroup $2\mathbb{Z}$, we have a short exact sequence

$$1 \to B_n' \to B_n^2 \to 2\mathbb{Z} \to 1.$$

By the right-exactness of the abelianization functor, we obtain a further short exact sequence

$$(B_n')^{ab} \to (B_n^2)^{ab} \to 2\mathbb{Z} \to 1.$$

Since $B_n'$ is perfect, we have $(B_n')^{ab} = 0$, and so the rightmost map is an isomorphism $(B_n^2)^{ab} \to 2\mathbb{Z}$. This shows that the abelianization of $B_n^2$ is cyclic, and that the kernel of the abelianization map $B_n^2 \to 2\mathbb{Z}$ is equal to $B_n'$. □

**Lemma 8.3.** *Let $n \geq 3$, and let $f$ be an element of $B_n$ that commutes with some half-twist $h$. Then the set of $B_n$-conjugates of $f$ is equal to the set of $B_n'$-conjugates of $f$. Both are equal to the set of $B_n^2$-conjugates of $f$.*

*Proof.* Let $g \in B_n$ and consider the conjugate $gfg^{-1}$ of $f$. Let $\ell$ denote the signed word length of $g$. Then $gh^{-\ell}$ lies in $B_n'$ and $(gh^{-\ell})f(gh^{-\ell})^{-1}$ is equal to $gfg^{-1}$. Since $B_n' \subseteq B_n^2$, the second statement follows. □

**Lemma 8.4.** *Let $n \geq 5$. The group $B_n'$ is the normal closure in $B_n'$ of $\sigma_1\sigma_3^{-1}$. Similarly, $B_n'$ is the normal closure in $B_n'$ of $\sigma_1\sigma_2^{-1}$.*

*Proof.* We already know that $B_n'$ is generated by the $B_n$-conjugates of $\sigma_1\sigma_3^{-1}$, and also that $B_n'$ is generated by the $B_n$-conjugates of $\sigma_1\sigma_2^{-1}$. The lemma thus follows from Lemma 8.3. □

The periodic braids $\alpha_1$ and $\alpha_2$ have signed word length equal to $n-1$ and $n$, respectively. We conclude that $\alpha_1 \in B_n^2$ if and only if $n$ is odd and that $\alpha_2 \in B_n^2$ if and only if $n$ is even. Also, we see that $z = \alpha_1^n$ lies in $B_n^2$ for all $n$.

**Lemma 8.5.** *Let $n \geq 5$. Let $\rho : B_n^2 \to B_n$ be a homomorphism. Assume that $\rho$ does not have cyclic image and that $\rho(z)$ is periodic. After possibly post-composing $\rho$ with a conjugation in $B_n$, the following statements hold.*

*(1) If $n$ is odd, then $\bar{\rho}(\alpha_1) = \bar{\alpha}_1^k$ with $k = \pm 1$.*
*(2) If $n$ is even, then $\bar{\rho}(\alpha_2) = \bar{\alpha}_2^k$ with $k = \pm 1$.*

*Proof.* We prove the second statement only; the proof of the first statement is similar. The proof follows the same outline as the proof of Proposition 4.1(2). As in that proof, we may use the classification of periodic elements in $\bar{B}_n$ in order to assume without loss of generality that either $\bar{\rho}_i(\alpha_2) = \bar{\alpha}_1^k$ or $\bar{\rho}_i(\alpha_2) = \bar{\alpha}_2^k$. Also, as in the same proof, it suffices to show that if $\bar{\rho}(\alpha_2)$ is equal to a power of $\bar{\alpha}_1^k$ or if $\bar{\rho}(\alpha_2)$ is equal to $\bar{\alpha}_2^k$ with $\gcd(k,n) \neq 1$ then $\bar{\rho}$ has cyclic image.

We first show that if $\bar{\rho}(\alpha_2)$ is equal to a power of $\bar{\alpha}_1^k$ then $\bar{\rho}$ has cyclic image. Suppose that $\bar{\rho}(\alpha_2) = \bar{\alpha}_1^k$. Because $\bar{\alpha}_1$ has order $n$ we may assume that $0 \leq k < n$. Since $\alpha_1^n = \alpha_2^{n-1}$ we have that

$$\bar{\rho}(\alpha_1^2)^{n/2} = \bar{\rho}(\alpha_2)^{n-1}.$$

Combining this with the equality $\bar{\rho}(\alpha_2) = \bar{\alpha}_1^k$, we conclude that

$$\bar{\rho}(\alpha_1^2)^n = \bar{\alpha}_1^{2k(n-1)}.$$

As in the proof of Proposition 4.1(2), we consider the composition

$$\bar{B}_n \xrightarrow{p} \mathbb{Z}/n(n-1) \to \mathbb{Z}/n.$$

We apply this composition to the last equality. The left-hand side maps to 0 and the right-hand side maps to $2k(n-1)^2 \equiv 2k$. Thus $k$ lies in $\{0, n/2\}$. In either case $\alpha_2^2$ lies in the kernel, and so by Lemma 4.2, we have that $\bar{\rho}$ has cyclic image, as desired.

We next show that if $\bar{\rho}_i(\alpha_2) = \bar{\alpha}_2^k$ with $\gcd(k,n) \neq 1$ then $\bar{\rho}$ has cyclic image. Since $\bar{\alpha}_2$ has order $n-1$ we may assume that $0 \leq k < n-1$. As in the proof of Proposition 4.1(2) we have that $(n-1)|kt$, where $t = (n-1)/\gcd(n-1,k)$. By the assumption that $\gcd(n-1,k) \neq 1$ we have that $0 < t < n-1$. We also have that $\alpha_2^t \in \ker \bar{\rho}$. By Lemma 4.2, the map $\bar{\rho}$ has cyclic image. This completes the proof. □

**Lemma 8.6.** *Assume that $n \geq 5$. Let $\rho : B_n \to B_m$ be a homomorphism. If $\bar{\rho}(\sigma_1^2)$ has empty canonical reduction system then $\bar{\rho}$ has cyclic image.*

*Proof.* Since $M(\bar{\rho}(\sigma_1^2))$ is empty, we have that $\bar{\rho}(\sigma_1^2)$ is either periodic or pseudo-Anosov. We consider the two cases separately. The proof follows the same outline as the proof of Lemma 7.3.

Assume that $\bar{\rho}(\sigma_1^2)$ is pseudo-Anosov. By Lemma 8.3, the braids $\sigma_i^2$ are pairwise conjugate in $B_n'$, hence in $B_n^2$. Because of this, the $\bar{\rho}(\sigma_i^2)$ are all pseudo-Anosov. The $\bar{\rho}(\sigma_{2i-1}^2)$ also commute pairwise, and so Lemma 5.2 implies that they all have equal centralizers. Since $n \geq 5$, each $\bar{\rho}(\sigma_{2i}^2)$ commutes with some $\bar{\rho}(\sigma_{2i-1}^2)$, and so it further follows that all of the $\bar{\rho}(\sigma_i^2)$ have the same centralizer in $\bar{B}_n$. In particular, they all commute.

Each $\bar{\rho}(\sigma_1\sigma_k^{-1})$ commutes with either $\bar{\rho}(\sigma_1^2)$ (when $k > 2$) or $\bar{\rho}(\sigma_4^2)$ (when $k = 2$). It follows that each $\bar{\rho}(\sigma_1\sigma_k^{-1})$ lies in the centralizer of $\bar{\rho}(\sigma_1^2)$ in $\bar{B}_n$ (which is equal to the centralizer of $\bar{\rho}(\sigma_1^2)$).

Gorin–Lin [32, p. 7] gave a finite presentation for $B_n'$. A consequence of their presentation is that $B_n'$ is generated by the braids $\{\sigma_i\sigma_1^{-1} \mid 2 \leq i \leq n-1\}$ (their generating includes elements $v$ and $w$, but their relations (1.16) and (1.20) show that these elements are products of the other generators). It follows that the group $B_n^2$ is generated by these and the $\sigma_i^2$.

The centralizer of $\bar{\rho}(\sigma_1^2)$ in $\bar{B}_n$ is abelian by Lemma 5.2.

We already showed that each of the generators for $B_n^2$ given in the previous paragraph lies in the centralizer of $\bar{\rho}(\sigma_1^2)$ in $\bar{B}_n$. By Lemma 5.2, this centralizer is abelian. Therefore, the image of $\bar{\rho}$ is abelian. It follows that the image of $\rho$ is abelian, hence it has cyclic image by Lemma 7.2.

Next, we assume that $\bar{\rho}(\sigma_1^2)$ is periodic. As in the proof of Lemma 7.3, we have that $\bar{\rho}(\sigma_1^2)$ and $\bar{\rho}(\sigma_3^2)$ generate a finite abelian subgroup of $\bar{B}_n$ that is conjugate to a a subgroup of either $\langle\bar{\alpha}_1\rangle$ or $\langle\bar{\alpha}_2\rangle$. Since $\bar{\rho}(\sigma_1^2)$ is conjugate to $\bar{\rho}(\sigma_3^2)$ in $B_n^2$ by Lemma 8.3, we must have $\bar{\rho}(\sigma_1^2) = \bar{\rho}(\sigma_3^2)$. In other words, $\bar{\rho}(\sigma_1\sigma_3^{-1})^2 = 1$. This implies that $\rho(\sigma_1\sigma_3^{-1})^2 = z^m$ for some $m$, hence that $z^m$ lies in the commutator subgroup of $B_n^2$. Since the commutator subgroup of $B_n^2$ is $B_n'$ (Lemma 8.2), we in particular have that $\sigma_1\sigma_3^{-1}$ lies in the commutator subgroup of $B_n^2$. Thus $\rho(\sigma_1\sigma_3^{-1}) = z^m$ lies in the commutator subgroup of $B_n$, which implies that $m = 0$. In other words, $\rho(\sigma_1\sigma_3^{-1})^2 = 1$. Since $B_n$ is torsion-free, we have that $\rho(\sigma_1\sigma_3^{-1}) = 1$. As above, it follows that $\rho$, hence $\bar{\rho}$, has cyclic image. □

The following lemma (and a proof)appears in the paper by the second and third authors [28, Lemma 3.2].

**Lemma 8.7.** *Let $n \geq 1$, let $c$ and $d$ be disjoint curves in $\mathbb{D}_n$ that surround exactly two marked points each, and suppose that the braid $(H_cH_d^{-1})^\ell$ has a $k$th root $f$. Then $\ell$ is divisible by $k$ and*

$$f = (H_cH_d^{-1})^{\ell/k}.$$

*Proof of Theorem 8.1.* The braid $z$ lies in $B_n^2$ for all $n$. As in the proof of Theorem 7.1 we consider three cases, according to whether $\bar{\rho}(z)$ is...

(1) pseudo-Anosov,
(2) periodic, or
(3) has non-empty canonical reduction system

*Case 1: $\bar\rho(z)$ is pseudo-Anosov.* By the same argument as in Proposition 5.1, we have that $\bar\rho$ has cyclic image. By the same argument as in Lemma 7.2, and applying Lemma 8.2, $\rho$ itself has cyclic image.

*Case 2: $\bar\rho(z)$ is periodic.* Assume that $\rho$ is not cyclic. By the same argument as in Lemma 7.2 (and applying Lemma 8.2), we also have that $\bar\rho$ is not cyclic.

We first claim that—up to post-composing $\rho$ with an automorphism of $B_n$—we have $\rho(\sigma_1^2) = H_{c_1}^\ell z^s$, where $c_1$ is a curve surrounding exactly two marked points, and $\ell$ and $m$ are integers. The argument is essentially the same as the argument given in Case 2 of the proof of Theorem 7.1 (minus the last paragraph), except with $\bar\rho(\sigma_1)$ replaced by $\bar\rho(\sigma_1^2)$, with Proposition 4.1 replaced by Lemma 8.5, with Lemma 7.3 replaced by Lemma 8.6, and—in the case of $n$ even—with $\alpha_1$ replaced by $\alpha_2$.

By Lemma 8.3, we have for each $1 \le i \le n-1$ that $\sigma_i^2$ is conjugate to $\sigma_1^2$ in $B_n^2$. Combining this with the previous claim, it follows that there exist curves $c_i$, each surrounding exactly two marked points, so that $\rho(\sigma_i^2) = H_{c_i}^\ell z^s$ for $1 \le i \le n-1$.

We claim that the $c_i$ are pairwise distinct. Suppose to the contrary that $c_j = c_k$. It follows that

$$\rho(\sigma_j\sigma_k^{-1})^2 = \rho(\sigma_j^2\sigma_k^{-2}) = (H_{c_j}^\ell z^s)(H_{c_k}^\ell z^s)^{-1} = 1.$$

Since $B_n$ is torsion-free, this implies $\rho(\sigma_j\sigma_k^{-1}) = 1$. By Lemma 8.3 we have that $\sigma_j\sigma_k^{-1}$ is conjugate in $B_n^2$ to either $\sigma_1\sigma_2^{-1}$ or $\sigma_1\sigma_3^{-1}$. It then follows from Lemma 8.4 that $\rho$ has cyclic image, contrary to assumption.

Our next claim is that $\ell$ is even and that $\rho(\sigma_1\sigma_4^{-1}) = (H_{c_1}H_{c_4}^{-1})^r$, where $\ell = 2r$. Since $\rho(\sigma_i^2) = H_{c_i}^\ell z^s$ we have

$$\rho(\sigma_1\sigma_4^{-1})^2 = \rho(\sigma_1^2\sigma_4^{-2}) = (H_{c_1}H_{c_4}^{-1})^\ell.$$

The claim now follows from Lemma 8.7.

We next claim that $\rho(\sigma_2\sigma_4^{-1}) = (H_{c_2}H_{c_4})^r$ and that $c_2$ is disjoint from $c_4$. As above, we also have $\rho(\sigma_2^2) = H_{c_2}^\ell z^s$. Since $\sigma_2^2$ commutes with $\sigma_4^2$, we have that $c_2$ is disjoint from $c_4$, which is the second statement of the claim. It follows that $H_{c_2}$ and $H_{c_4}$ commute, and so

$$\rho(\sigma_2\sigma_4^{-1})^2 = \rho(\sigma_2^2\sigma_4^{-2}) = \rho(\sigma_2^2)\rho(\sigma_4^{-2}) = (H_{c_2}H_{c_4}^{-1})^\ell.$$

It follows that $\rho(\sigma_2\sigma_4^{-1})$ is equal to a square root of $(H_{c_2}H_{c_4}^{-1})^\ell$. The claim follows now from Lemma 8.7.

Our next claim is that, up to post-composing $\rho$ with an automorphism of $B_n$, we have that $\ell = 2r = 2$ and that $i(c_1, c_2) = 2$. Since $\sigma_4$ commutes with both $\sigma_1$ and $\sigma_2$, and since $\sigma_1$ and $\sigma_2$ satisfy the braid relation, the braids $\sigma_1\sigma_4^{-1}$ and $\sigma_2\sigma_4^{-1}$ also satisfy the braid relation. It follows that the images of these braids under $\rho$ satisfy the braid relation. Using the descriptions of these images given in the previous two claims (including the fact that $i(c_2, c_4) = 0$), it follows that $H_{c_1}^r$ and $H_{c_2}^r$ satisfy the braid relation. As in Case 2 of the proof of Theorem 7.1, the claim then follows from the result of Bell and the second author cited there.

We now claim that, up to post-composing $\rho$ by an automorphism of $B_n$, we have $\rho(\sigma_i^2) = \sigma_i^2 z^s$. By the same argument used to prove Lemma 8.3, each ordered pair $(\sigma_i^2, \sigma_{i+1}^2)$ is conjugate in $B_n^2$ to the ordered pair $(\sigma_1^2, \sigma_2^2)$. Since $i(c_1, c_2) = 2$, it follows that $i(c_i, c_{i+1}) = 2$ for all $1 \le i \le n-1$. For $1 \le i, j \le n-1$ the braids $\sigma_{2i}$ and $\sigma_{2j}$ commute, and so $i(c_{2i}, c_{2j}) = 0$. Combining the last two sentences, we may conclude the claim as in the proof of Theorem 7.1.

Finally, we show that $\rho$ is a transvection of the standard inclusion. The group $B_n^2$ is generated by $B_n'$ together with the $\sigma_i^2$. By the last claim, there is an $s$ so that $\rho(\sigma_i^2) = \sigma_i^2 z^s$

for all $i$. Since the signed word length of each element of $B_n$ is 0, it suffices to show that $B_n'$ is fixed element-wise by $\rho$. Gorin–Lin [32, p. 7] gave a finite presentation for $B_n'$. A consequence of their presentation is that $B_n'$ is generated by the braids $\{\sigma_i\sigma_1^{-1} \mid 2 \le i \le n-1\}$ (their generating includes elements $v$ and $w$, but their relations (1.16) and (1.20) show that these elements are products of the other generators). To complete the proof we will show that $\rho(\sigma_i\sigma_j^{-1}) = \sigma_i\sigma_j^{-1}$ for all $1 \le i, j \le n-1$.

If $|i-j| > 1$ then

$$\rho(\sigma_i\sigma_j^{-1})^2 = \rho(\sigma_i^2\sigma_j^{-2}) = \sigma_i^2\sigma_j^{-2} = (\sigma_i\sigma_j^{-1})^2.$$

It now follows from Lemma 8.7 that $\rho(\sigma_i\sigma_j^{-1}) = \sigma_i\sigma_j^{-1}$, as desired.

If $|i-j| = 1$, we choose some $k$ with $|i-k| > 1$ and $|j-k| > 1$. We then have

$$\rho(\sigma_i\sigma_j^{-1}) = \rho(\sigma_i\sigma_k^{-1})\rho(\sigma_k\sigma_j^{-1}) = (\sigma_i\sigma_k^{-1})(\sigma_k\sigma_j^{-1}) = \sigma_i\sigma_j^{-1},$$

as desired. This completes the proof of Case 2.

*Case 3: $\bar\rho(z)$ has nonempty canonical reduction system.* The argument for this case is essentially the same as for Case 3 of the proof of Theorem 7.1, once we account for the fact that the commutator subgroup of $B_n^2$ is equal to $B_n'$ (Lemma 8.2). This completes the proof. □

## 9. Cablings

As in Section 2, a map $\rho : B_n \to B_{2n}$ is a *2-fold cabling map* if there is a multicurve $M$ in $\mathbb{D}_{2n}$ that has $n$ components and so that the action of $\rho(B_n)$ on the set of components of $M$ is standard. We refer to $M$ as the *cabling multicurve* for $\rho$. We note that each component of $M$ surrounds exactly two marked points in $\mathbb{D}_{2n}$. The goal of this section is to prove the following proposition.

**Proposition 9.1.** *Let $n \ge 2$ and let $\rho : B_n \to B_{2n}$ be a 2-fold cabling map with cabling multicurve $M$. Then $\rho$ is equivalent to one of the standard $k$-twist cabling maps.*

We begin with two lemmas. In both statements the standard curve $c_i$ is the curve in $\mathbb{D}_{2n}$ with the property that $H_{c_i} = \sigma_i$. The set of standard curves $C = \{c_1, c_3, \dots, c_{2n-1}\}$ is a multicurve. By Lemma 6.2 we have associated to $C$ a homomorphism $\Pi_e : \mathrm{Stab}_{B_{2n}}(C) \to B_n$.

**Lemma 9.2.** *Let $n \ge 2$ and let $\rho : B_n \to B_{2n}$ be a 2-fold cabling map with cabling multicurve $M$. Then, up to replacing $\rho$ by an equivalent homomorphism, we have that*

*(1) $\rho$ is a 2-fold cabling map with cabling multicurve $C = \{c_1, c_3, \dots, c_{2n-1}\}$, and*
*(2) $\rho$ is a section of the map $\Pi_e$ associated to $C$.*

*Proof.* Up to modifying $\rho$ by an inner automorphism of $B_{2n}$, we may assume (by the change of coordinates principle [18, Section 1.3]) that $M$ is the set of standard curves $C = \{c_1, c_3, \dots, c_{2n-1}\}$ (meaning that $H_{c_i}$ is $\sigma_i$ for each $i$). By Lemma 6.2, we have a split short exact sequence

$$1 \to \prod_{i=1}^{n} B_2 \to \mathrm{Stab}_{B_{2n}}(C) \xrightarrow{\Pi_e} B_n \to 1.$$

The composition $\Pi_e \circ \rho$ is a homomorphism from $B_n$ to $B_n$.

By the previous paragraph, the post-composition of $\Pi_e \circ \rho$ by the projection $B_n \to S_n$ is standard. In particular $\Pi_e \circ \rho$ does not have cyclic image. It then follows from Theorem 7.1 that there is an automorphism $\alpha$ of $B_n$ and a $z^k \in Z(B_n)$ so that $\alpha \circ (\Pi_e \circ \rho)^{z_k}$ is equal to the identity. We have that $(\Pi_e \circ \rho)^{z_k}$ is equal to $\Pi_e \circ \rho^{z_k}$ (the $z$ is an element of $B_n$ in the first expression and is an element of $B_{2n}$ in the second). Combining the last two sentences, we have that $\alpha \circ \Pi_e \circ \rho^{z_k}$ is equal to the identity.

Since automorphisms of $B_n$ are induced by homeomorphism of $\mathbb{D}_n$, there is an automorphism $\tilde{\alpha}$ of $B_{2n}$ so that $\alpha \circ \Pi_e \circ \rho^{z_k}$ is equal to $\Pi_e \circ \tilde{\alpha} \circ \rho^{z_k}$ (the homeomorphism of $\mathbb{D}_{2n}$ inducing $\tilde{\alpha}$ necessarily stabilizes $C$). In particular the latter is equal to the identity. Since $\rho' = \tilde{\alpha} \circ \rho^{z_k}$ is equivalent to $\rho$ and it is a section of $\Pi_e$, we have proven the second statement of the lemma. For the first statement, the only thing left to prove is that $C$ is all of $M$. But this is true since no multicurve in $\mathbb{D}_n$ is preserved by all of $B_n$, and since $\Pi_e \circ \rho$ is the identity. □

In what follows, let $A$ denote the kernel of $\Pi_e : \mathrm{Stab}_{B_{2n}}(C) \to B_n$. As per Section 6, the group $A$ is isomorphic to $B_2 \times \cdots \times B_2 \cong \mathbb{Z}^n$. The map $\beta$ is the one described in Section 6; in the present situation, $\beta$ is nothing other than the standard 0-twist cabling map $B_n \to B_{2n}$.

From the short exact sequence

$$1 \to PB_n \to B_n \to S_n \to 1$$

there is an action of $S_n$ on $H^1(PB_n; \mathbb{Z}^n)$. In the statement of the next lemma, $H^1(PB_n; \mathbb{Z}^n)^{S_n}$ denotes the subgroup of $S_n$-invariant elements of $H^1(PB_n; \mathbb{Z}^n)$.

**Lemma 9.3.** *Let $n \geq 2$ and let $\Pi_e : \mathrm{Stab}_{B_{2n}}(C) \to B_n$ be the map associated to the standard multicurve $C$. Then there is an injective map*

$$\{A\text{-conjugacy classes of sections of } \Pi_e\} \to H^1(PB_n; \mathbb{Z}^n)^{S_n}$$

*given by*

$$[s] \mapsto (g \mapsto s(g)\beta(g)^{-1}).$$

*Proof.* From the short exact sequence

$$1 \to A \to \mathrm{Stab}_{B_{2n}}(C) \xrightarrow{\Pi_e} B_n \to 1$$

we have a bijection

$$H^1(B_n; A) \leftrightarrow \{A\text{-conjugacy classes of sections of } \Pi_e\}$$

(see [9, Ch. IV.2]). There is a preferred section of $\Pi_e$, namely $\beta$; the $2n$-stranded braid $\beta(\sigma_i)$ is obtained from the $n$-stranded braid $\sigma_i$ by replacing each strand by a 2-stranded trivial braid. In the given bijection, we regard $A$ as a $B_n$-module via the rule

$$g \cdot a = \beta(g) a \beta(g)^{-1}.$$

Under this action, $B_n$ permutes the generators of $A$ according to the standard homomorphism $B_n \to S_n$.

The group $H^1(B_n; A)$ is the abelian group of crossed homomorphisms $B_n \to A$ modulo the subgroup of principal crossed homomorphisms (see [9, Ch. IV.2]). In these terms, the above bijection is defined by sending the $A$-conjugacy class of a section $s$ to the cohomology class of the crossed homomorphism $d_s$ defined by

$$d_s(g) = s(g)\beta^{-1}(g).$$

The inclusion map $PB_n \to B_n$ induces a restriction homomorphism

$$\Xi : H^1(B_n; \mathbb{Z}^n) \to H^1(PB_n; \mathbb{Z}^n)^{S_n}$$

Since $PB_n$ acts trivially on $A$, we have that $H^1(PB_n; \mathbb{Z}^n)^{S_n}$ is the group of $S_n$-equivariant homomorphisms $H_1(PB_n; \mathbb{Z}) \to A$.

To prove the lemma, we need to show that $\Xi$ is injective. The 5-term exact sequence in cohomology [9, Ch. VII.6] associated to the short exact sequence

$$1 \to PB_n \to B_n \to S_n \to 1$$

contains the sequence

$$H^1(S_n; \mathbb{Z}^n) \to H^1(B_n; \mathbb{Z}^n) \xrightarrow{\Xi} H^1(PB_n; \mathbb{Z}^n)^{S_n}.$$

By Shapiro's lemma, we have $H^1(S_n; \mathbb{Z}^n) \cong H^1(S_{n-1}; \mathbb{Z}) \cong 0$, because $\mathbb{Z}^n$ is isomorphic to the co-induced representation $\mathrm{CoInd}_{S_{n-1}}^{S_n} \mathbb{Z}$. It follows that $\Xi$ is injective, as desired. □

*Proof of Proposition 9.1.* By Lemma 9.2, we may assume that

(1) the cabling multicurve for $\rho$ is $C = \{c_1, c_3, \ldots, c_{2n-1}\}$, and
(2) $\rho$ is a section of the map $\Pi_e$ associated to $C$.

By Lemma 9.3, $\rho$ corresponds to an element $d_\rho$ of $H^1(PB_n; \mathbb{Z}^n)^{S_n}$.

Let us denote the standard $k$-twist cabling by $\rho_k$. We will produce an inner automorphism $\alpha$ of $B_{2n}$ and a $t$ in the centralizer of $\rho(B_n)$ so that $\alpha \circ \rho^t$ is equal to $\rho_k$ for some $k$.

With respect to the semi-direct product decomposition for $\mathrm{Stab}_{B_{2n}}(C)$ from Lemma 6.1 we can write

$$\rho_k(\sigma_i) = (\sigma_i, (0, \ldots, 0, k, 0, \ldots, 0))$$

where $k$ lies in the $i$th entry. Since $\rho(\sigma_i)$ also lies in $\mathrm{Stab}_{B_{2n}}(C)$, we may write it as

$$\rho(\sigma_i) = (\sigma_i, (k_{i1}, \ldots, k_{in}))$$

for some integers $k_{i1}, \ldots, k_{in}$. We will choose $\alpha$ and $t$ in order to modify $\rho(\sigma_i)$ so that the vector $(k_{i1}, \ldots, k_{in})$ matches the vector $(0, \ldots, 0, k, 0, \ldots, 0)$.

We claim that there exist integers $x$ and $y$ such that

$$d_\rho(\sigma_i^2) = (x, \ldots, x, y, y, x, \ldots, x) \text{ for each } 1 \leq i \leq n-1.$$

Let $a_1, \ldots, a_n$ be integers such that $d_\rho(\sigma_1^2) = (a_1, \ldots, a_n)$. To prove the claim, we first show that $a_i = a_{i+1}$ and all other $a_j$ are equal. The equivariance of $d_\rho$ implies that

$$d_\rho(\sigma_i \sigma_1^2 \sigma_i^{-1}) = \sigma_i \cdot d_\rho(\sigma_1^2) = \sigma_i \cdot (a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n).$$

On the other hand, since $\sigma_1$ commutes with $\sigma_i$ for $i \neq 2$, we have that $d_\rho(\sigma_i \sigma_1^2 \sigma_i^{-1}) = d_\rho(\sigma_1^2)$ for each $i \neq 2$. Thus for each $i \neq 2$ we have that

$$(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n)$$

and therefore that $a_i = a_{i+1}$ for $i \neq 2$, as desired. This shows that there are integers $x, y$ such that $d_\rho(\sigma_1^2) = (y, y, x, \ldots, x)$. The claim now follows from the equivariance of $d_\rho$.

We next claim that $2k_{ij} = x$ for all $i \notin \{j, j+1\}$ and $k_{jj} + k_{j+1,j} = y$ for all $j$. We have

$$d_\rho(\sigma_j^2) = \rho(\sigma_1^2)\beta(\sigma_1^2)^{-1} = (2k_{1j}, \ldots, 2k_{j-1,j}, k_{jj} + k_{j+1,j}, k_{jj} + k_{j+1,j}, 2k_{j+2,j}, \ldots, 2k_{n,j})$$

By the previous claim, we have $2k_{ij} = x$ for all $i \notin \{j, j+1\}$ and $k_{jj} + k_{j+1,j} = y$ for all $j$, as desired.

Let $t = (\sigma_1 \sigma_3 \cdots \sigma_{2n-1})^{-x/2}$. We may compute that

$$\rho^t(\sigma_i) = (\sigma_i, (0, \ldots, 0, k_{ii} - x/2, k_{i+1,i} - x/2, 0, \ldots, 0))$$

where the two nonzero entries in the $\mathbb{Z}^n$ factor are in the $i$th and $(i+1)$st coordinates.

Let $\alpha$ denote the inner automorphism corresponding to conjugation by

$$h = \sigma_1^{m_1} \sigma_3^{m_2} \cdots \sigma_{2n-1}^{m_n}.$$

Then $\alpha \circ \rho^t(\sigma_i)$ is the braid

$$\rho^t(\sigma_i) = (\sigma_i, (0, \ldots, 0, m_i - m_{i+1} + k_{ii} - x/2, m_{i+1} - m_i + k_{i+1,i} - x/2, 0, \ldots, 0))$$

As above, if we want $\alpha \circ \rho^t(\sigma_i)$ to agree with $\rho_k(\sigma_i)$ we need to choose $\alpha$ so that

$$m_{i+1} - m_i + k_{i+1,i} - x/2 = 0.$$

for all $i$ and so that the quantity $m_{i+1} - m_i + k_{i+1,i} - x/2$ is independent of $i$. The system of equations $\{m_{i+1} - m_i + k_{i+1,i} - x/2 = 0\}$ does indeed admit (infinitely many) integer solutions. For any such solution, we then have that

$$m_i - m_{i+1} + k_{ii} - x/2 = (k_{i+1,i} - x/2) + k_{i,i} - x/2 = (k_{i+1,i} + k_{ii}) - x = y - x$$

is independent of $i$, as desired. This completes the proof of the proposition. □

## 10. Transvections of the standard homomorphisms

In this section we first give complete descriptions of the centralizers of the images of the standard homomorphisms $B_n \to B_{2n}$. Afterwards, we use these descriptions to show that the standard homomorphisms $B_n \to B_{2n}$ are pairwise inequivalent.

We begin with the descriptions of the centralizers of the images of the standard homomorphisms $B_n \to B_{2n}$. As discussed in the introduction, every transvection of one of the standard homomorphisms is given by such an element. We do not give the proofs that the centralizers are as stated; in each case the centralizer is determined by the basic theory surrounding the Nielsen–Thurston classification theorem and the theory of canonical reduction systems.

Consider first the trivial map $\rho : B_n \to B_{2n}$. This this case the centralizer of $\rho(B_n)$ is the entire group $B_{2n}$.

Now consider the inclusion map $\rho : B_n \to B_{2n}$. In this case $\rho$ is induced by an embedding $\mathbb{D}_n \to \mathbb{D}_{2n}$. The centralizer of $\rho(B_n)$ consists of exactly those elements that have a representative with support outside of the image of $\mathbb{D}_n$.

Next consider the diagonal embedding $\rho : B_n \to B_{2n}$. Here $\rho$ is induced by an embedding $\mathbb{D}_n \sqcup \mathbb{D}_n \to \mathbb{D}_{2n}$. The centralizer of $\rho(B_n)$ is generated by three elements, namely, the Dehn twists about the boundaries of the images of the two copies of $\mathbb{D}_n$ and an element of $B_{2n}$ that swaps the two disks.

Now consider the flip diagonal embedding $\rho : B_n \to B_{2n}$. In this case the centralizer of $\rho(B_n)$ is again generated by three elements, this time the Dehn twists about the boundaries of the images of the two copies of $\mathbb{D}_n$ and the Dehn twist about the boundary of $\mathbb{D}_{2n}$ (namely, $z$).

Finally, we consider the $k$-twist cabling map $\rho : B_n \to B_{2n}$. In this case the centralizer of $\rho(B_n)$ is a free abelian group of rank 2. It is generated by $z$ and the product of the $n$ half-twists about the cabling curves $c_1, c_3, \ldots, c_{2n-1}$. This product is nothing other than $\sigma_1\sigma_3 \cdots \sigma_{2n-1}$.

**Proposition 10.1.** *Let $n \geq 5$. The standard homomorphisms $B_n \to B_{2n}$ are pairwise inequivalent.*

*Proof.* To begin, we note that the trivial map is not equivalent to the inclusion, diagonal inclusion, flip-diagonal inclusion, or any of the $k$-twist cablings, since a homomorphism is equivalent to the trivial map if and only if its image is cyclic.

The $k$-twist cabling maps are not equivalent to the inclusion map, the diagonal inclusion map, or the flip diagonal inclusion map, since the latter three do not have any reducing multicurves that surround exactly two marked points, and because any homomorphism equivalent to a $k$-twist cabling map must have such a reducing curve. So it remains to distinguish the $k$-twist cabling maps from each other and the inclusion map, the diagonal inclusion map, and the flip diagonal inclusion map from each other.

Let us show that the flip-diagonal inclusion map is not equivalent to the inclusion map or to the diagonal inclusion map. If $\rho$ is equivalent to the flip diagonal inclusion map, then there are disjoint disks $\Delta_1$ and $\Delta_2$ in $\mathbb{D}_{2n}$ that contain $n$ marked points each, and so that $\rho(\sigma_1)$ is the product of a positive half-twist in $\Delta_1$, a negative half-twist in $\Delta_2$, and a mapping class supported outside of $\Delta_1$ and $\Delta_2$. Any such braid is not equal to the image of $\sigma_1$ under the diagonal inclusion map or the inclusion map, as desired.

We now show that the inclusion map is not equivalent to the diagonal inclusion map. If $\rho$ is equivalent to the inclusion map, then there is a disk $\Delta$ that contains $n$ marked points and braid $f$ supported in the exterior of $\Delta$ so that for each $i$, the braid $\rho(\sigma_i)$ is the product of $f$ with a (possibly negative) half-twist in $\Delta$. Since $f$ does not depend on $i$, we see that $\rho$ cannot be equal to the diagonal inclusion map.

It remains to distinguish the $k$-twist cabling maps from each other. Suppose that $\rho : B_n \to B_{2n}$ is a homomorphism that is equivalent to a $k$-twist cabling map $\rho_k$. Let $\rho_\ell$ be any other $k$-twist cabling map. We will show that $\rho$ is not equal to $\rho_\ell$. By the classification of transvections from Section 10, we must have that $\rho = \alpha \circ \rho_k^t$, where $\alpha$ is an automorphism of $B_{2n}$ and $t$ is a product of two elements, namely, a power of $z$, and a power $\tau^p$ of $\tau = \sigma_1\sigma_3 \cdots \sigma_{2n-1}$. Let $\Pi_e : \mathrm{Stab}_{B_{2n}}(C) \to B_n$ be the map from Section 6. Since $\rho_k$ and $\rho_\ell$ are both sections for $\Pi_e$, it follows that $\alpha$ is the inner automorphism corresponding to some product $\sigma_1^{k_1}\sigma_3^{k_3} \cdots \sigma_{2n-1}^{k_{2n-1}}$. By considering the images of the $\sigma_i$ under $\rho$ and comparing to the $\rho_\ell(\sigma_i)$ we conclude that the $k_i$ are all equal. In other words, $\alpha$ is the inner automorphism corresponding to some power of $\tau$. But $\tau$ lies in the centralizer of $\rho(B_n)$, and so we may replace $\alpha$ by the trivial automorphism. Thus we have $\rho_\ell = \rho^{\tau^p}$ for some integer $p$. It follows that $p = 0$ and $k = \ell$, as desired. This completes the proof of the proposition. □

## 11. Proof of the main theorem

In this section we prove Theorem 1.1 using both Theorems 7.1 and 8.1.

*Proof of Theorem 1.1.* By Proposition 10.1, the standard homomorphisms $B_n \to B_{2n}$ are pairwise inequivalent. It remains to show that any homomorphism $B_n \to B_{2n}$ is equivalent to a standard homomorphism. We fix some $n \geq 5$.

To this end, we will prove by strong induction on $m$ the following statement: for $m \leq 2n$, every homomorphism $B_n \to B_m$ is equivalent to one of the standard homomorphisms, and in particular if $m < 2n$ then it is equivalent to either the trivial map or to the inclusion map. We will then argue that the standard homomorphisms $B_n \to B_{2n}$ are mutually inequivalent.

The base case is $m = n$, which is Theorem 7.1. Therefore, we may assume that $n < m \leq 2n$.

Denote the canonical reduction system of $\rho(z)$ by $M$. It follows from Corollary 4.3 and Proposition 5.1 that if $M$ is empty then the induced map $\bar{\rho}$ has cyclic image, and then from Lemma 7.2 that $\rho$ has cyclic image (hence is equivalent to the trivial map). Thus, we may henceforth assume that $M$ is non-empty.

As in the proof of Theorem 7.1, the map $\rho$ induces an action of $B_n$ on the set of components of $M$. We may write $M$ as a union of multicurves $M_p$, where each component of $M_p$ is a curve surrounding exactly $p$ marked points. The action of $B_n$ further restricts to an action on each $M_p$. Each $M_p$ has at most $m/p$ components. Since $m \leq 2n$ and $p \geq 2$, each $M_p$ has at most $n$ components.

For each $p$, the action of $B_n$ on the set of components of $M_p$ factors through a homomorphism $\tau_p : B_n \to B_k$ where $k = |M_p|$ and the standard projection $B_k \to S_k$; the map $\tau_p$ is obtained from $\rho$ by collapsing the disks bounded by $M_p$ to marked points and by forgetting all marked points not lying in the interior of a component of $M_p$. Since $k < m \leq 2n$, it follows from induction that $\tau_p$ is equivalent to either the trivial map or the identity map.

If there is a $p$ so that $\tau_p$ is equivalent to the identity map, then $p = 2$ and the action of $\rho(B_n)$ on the set of components of $M_2$ is standard. By Proposition 9.1, $\rho$ is equivalent to one of the standard $k$-twist cabling maps.

If all of the $\tau_p$ are equivalent to the trivial map, it follows that the action of $\rho(B_n)$ on the set of components of each $M_p$, hence on the set of components of $M$, is cyclic. We may assume henceforth that this is the case.

Let $P$ be the largest number so that $M_P$ is nonempty. If $P < n$, then, by the same argument as the one used in Case 3 of the proof of Theorem 7.1, it follows that $\rho$ has cyclic image. So we may henceforth assume that $P \geq n$. We now complete the proof in three separate cases:

(1) $M_P$ has one component.
(2) $M_P$ has two components, both fixed by $\rho$.
(3) $M_P$ has two components, interchanged by the action of $\rho$.

*Case 1: $M_P$ has a single component.* The argument here is based on the argument for Case 3 in the proof of Theorem 7.1. The main difference is that we must use our strong inductive hypothesis instead of Lin's theorem.

Since $M_p$ has a single component, we may apply the interior/exterior decomposition from Section 6. Let $\rho_i$ and $\rho_e$ be the corresponding interior and exterior components of $\rho$.

We claim that $\rho_e$ has cyclic image. By construction, the canonical reduction system of $\rho_e(z)$ is empty. If $m - P + 1$ is not equal to $n$, then we may conclude as above that $\rho_e$ has cyclic image. If $m - P + 1 = n$ then we know by Theorem 7.1 that $\rho_e$ has cyclic image or is equivalent to the identity map. But since $\rho_e(B_n)$ lies in $B_{m-P,1} \subsetneq B_{m-P+1} = B_n$, it must be that $\rho_e$ has cyclic image, as desired.

By strong induction, $\rho_i$ is equivalent to the standard inclusion $B_n \to B_P$. Since $\rho_e$ has cyclic image, we may assume, after replacing $\rho$ with an equivalent homomorphism, that $\rho_e$ is the identity (modify by the appropriate transvection). It follows from Lemma 6.1 that $\rho$ is equal to the post-composition of $\rho_i$ with the inclusion $B_P \to B_m$. In particular, $\rho$ is equivalent to the standard inclusion.

*Case 2: $M_P$ has two components, both fixed by $\rho$.* Since the components of $M_P$ are both fixed by $\rho$, we may apply the interior/exterior decomposition from Section 6.

The map $\rho_e$ has cyclic image, since its image is the cyclic group $B_2$. In fact the image must lie in $B_2^2$ since the components of $M_P$ are both fixed. Therefore, after modifying $\rho$ by a transvection, we may assume that $\rho_e$ is trivial.

The interior component of $\rho_i$ is

$$\rho_i : B_n \to B_n \times B_n.$$

It follows from Corollary 7.4 and the fact that each element of $\mathrm{Aut}(B_n)$ is induced by a homeomorphism of $\mathbb{D}_n$ that, up to post-composing $\rho$ by an automorphism of $B_{2n}$, each of the two components of $\rho_i$ is a transvection of either the identity map, the inversion map, or the trivial map, and further that at most one of the factors is the inversion map. It now follows from Lemma 6.1 that $\rho$ is equivalent to either the trivial homomorphism, the diagonal inclusion, or the flip-diagonal inclusion.

*Case 3: $M_P$ has two components, interchanged by the action of $\rho$.* We begin with some setup. In the present case Lemma 6.2 gives the semi-direct product decomposition

$$\mathrm{Stab}_{B_{2n}}(M_P) \cong B_2 \ltimes (B_n \times B_n).$$

where the generator $\sigma_1$ for $B_2 \cong \mathbb{Z}$ acts on $B_n \times B_n$ by interchanging the factors. In terms of this decomposition, the assumption that $\rho(B_n)$ acts nontrivially on the components of $M_P$ translates to the fact that $\rho(B_n)$ projects to a subgroup of $B_2$ that is not contained in $B_2^2$, and moreover that each $\rho(\sigma_i)$ projects to $\sigma_1^\ell \in B_2$ with $\ell$ odd (that $\ell$ is independent of $i$ follows from Lemma 8.3).

For $1 \le i \le n-1$, let $c_i$ and $c_i'$ denote the canonical reduction systems for $\sigma_i$ and $\sigma_{n+i}$, respectively (each canonical reduction system is a single curve). These are the standard curves in the interior of $m_1$ and $m_2$, each surrounding consecutive marked points.

Since $\rho(B_n)$ maps to $\mathrm{Stab}_{B_{2n}}(M_P)$ by assumption, we may write elements of $\rho(B_n)$ in terms of the semidirect product decomposition. So elements of $\rho(B_n)$ will be written as $(\sigma_1^k, (\alpha, \beta))$, where $k$ is an integer, and $\alpha$ and $\beta$ are elements of $B_n$.

Consider the restriction $\rho' = \rho|B_n^2$. The image of $\rho'$ lies in

$$\mathrm{Fix}_{B_{2n}}(M) \cong B_2^2 \times (B_n \times B_n).$$

This group has index 2 in $\mathrm{Stab}_{B_{2n}}(M)$. As in Section 6, we may post-compose $\rho'$ with the projections to $B_2^2$ and $B_n \times B_n$, and we denote the resulting homomorphisms by $\rho_e'$ and $\rho_i'$. We may further post-compose $\rho_i'$ with the projections to the two factors of $B_n \times B_n$ in order to obtain homomorphisms $\rho_{i,1}'$ and $\rho_{i,2}'$.

By Theorem 8.1, we may post-compose $\rho$ with an inner automorphism of $B_{2n}$ so that each of $\rho_{i,1}'$ and $\rho_{i,2}'$ is of one of the following types: cyclic, a transvection of the inclusion map, or a transvection of the (restriction of the) inversion map. We may further post-compose $\rho$ with an automorphism of $B_{2n}$ (either the identity or the inversion map) so that $\rho_{i,1}'$ is either cyclic or a transvection of the inclusion map.

We now begin the proof in earnest. We proceed in five steps.

*Step 1.* Both $\rho_{i,1}'$ and $\rho_{i,2}'$ have cyclic image or both do not.

*Step 2.* If $\rho_{i,1}'$ and $\rho_{i,2}'$ have cyclic image, then $\rho$ does.

*Step 3.* We may modify $\rho$ so $\rho_{i,1}'$ and $\rho_{i,2}'$ are both transvections of the inclusion map.

*Step 4.* We have $\rho(\sigma_i) = (\sigma_1^\ell, (\sigma_i z^{k_1'}, \sigma_i z^{k_2'}))$ for $k_1', k_2'$ and odd $\ell$.

*Step 5.* The map $\rho$ is equivalent to the diagonal inclusion map.

We now treat each of the five steps in turn.

*Step 1.* We claim that $\rho_{i,2}'$ has cyclic image if and only if $\rho_{i,1}'$ does. By the previous paragraph, both maps are either cyclic or injective. By symmetry, it suffices to show one implication. Suppose that $\rho_{i,1}'$ has cyclic image. Let $f$ be a nontrivial element of its kernel. It follows from the semi-direct product decomposition for $\mathrm{Stab}_{B_{2n}}(M)$ and the fact that $\rho(\sigma_1)$ maps to an element of $B_2 \setminus B_2^2$ that $\sigma_1 f \sigma_1^{-1}$ lies in the kernel of $\rho_{i,2}'$.

*Step 2.* If $\rho_{i,1}'$ and $\rho_{i,2}'$ both have cyclic image, then by the direct product decomposition for $\mathrm{Fix}_{B_{2n}}(M)$ it follows that $\rho'$ has cyclic image. By Lemma 8.2, $\rho'|B_n'$ is trivial. Since $\rho|B_n'$ is equal to $\rho'|B_n'$ it follows that $\rho$ has cyclic image. Therefore, we may henceforth assume that

$\rho'_{i,1}$ is a transvection of the identity map and that $\rho'_{i,2}$ is a transvection of either the identity map or the inversion map.

*Step 3.* We claim that $\rho'_{i,2}$ is a transvection of the inclusion map; in other words $\rho'_{i,2}$ is not a transvection of the inversion map. Since $\rho'_{i,1}$ is a transvection of the inclusion map and since $\rho'_{i,2}$ is a transvection of either the inclusion map or the inversion map, it follows that $\rho(\sigma_1^2) = \rho'(\sigma_1^2)$ is of the form

$$(\sigma_1^{2\ell}, (\sigma_1^2 z^{k_1}, \sigma_1^{2\epsilon} z^{k_2}))$$

where $\epsilon \in \{\pm 1\}$. Since $\rho(\sigma_1)$ commutes with $\rho(\sigma_1^2)$, the former preserves the canonical reduction system of the latter. Since $c_1$ and $c_1'$ are the only curves in the canonical reduction system of $\rho(\sigma_1^2)$ that surround exactly two marked points, and since $\rho(\sigma_1)$ permutes the two components of $M_P$, it follows that $\rho(\sigma_1)$ interchanges $c_1$ and $c_1'$. Again using the fact that $\rho(\sigma_1)$ commutes with $\rho(\sigma_1^2)$, it follows that $\epsilon = 1$, whence the claim.

*Step 4.* Our next claim is that for $1 \leq i \leq n-1$ we have $\rho(\sigma_i) = (\sigma_1^\ell, (\sigma_i z^{k_1'}, \sigma_i z^{k_2'}))$ for some integers $k_1'$ and $k_2'$ (and where $\ell$ is the odd integer given above). It is enough (by the Alexander method [18, Proposition 2.8]) to check that the left and right-hand sides of the formula have the same actions on the curves $m_1, m_2, c_1, \ldots, c_{n-1}, c_1', \ldots, c_{n-1}'$. Since $\rho'_{i,1}$ and $\rho'_{i,2}$ are both transvections of the inclusion map, we have

$$\rho(\sigma_i)\rho(\sigma_j^2)\rho(\sigma_i)^{-1} = \rho(\sigma_i\sigma_j^2\sigma_i^{-1}) = \left(\sigma_1^{2\ell}, (\sigma_i\sigma_j^2\sigma_i^{-1}z^{k_1}, \sigma_i\sigma_j^2\sigma_i^{-1}z^{k_2})\right).$$

There are two curves of the canonical reduction system for $\rho(\sigma_i)\rho(\sigma_j^2)\rho(\sigma_i^{-1})$ that surround exactly two marked points, namely, $\rho(\sigma_i)(c_j)$ and $\rho(\sigma_i)(c_j')$. Similarly, there are two curves of the canonical reduction system of $(\sigma_1^{2\ell}, (\sigma_i\sigma_j^2\sigma_i^{-1}z^{k_1}, \sigma_i\sigma_j^2\sigma_i^{-1}z^{k_2}))$ that surround exactly two marked points, namely, $\sigma_i(c_j)$ and $\sigma_{n+i}(c_j')$. It follows from the equality of the two braids that $\rho(\sigma_i)$ maps the set $\{c_j, c_j'\}$ to the set $\{\sigma_i(c_j), \sigma_{n+i}(c_j')\}$. Moreover, since $\rho(\sigma_i)$ interchanges the components of $M_P$ it follows that

$$\rho(\sigma_i)(c_j) = \sigma_{n+i}(c_j'), \text{ and}$$
$$\rho(\sigma_i)(c_j') = \sigma_i(c_j).$$

Since $\ell$ is odd, this agrees with the action of $(\sigma_1^\ell, (\sigma_i z^{k_1'}, \sigma_i z^{k_2'}))$ (since $(1, (z, 1))$, $(1, (1, z))$, and even powers of $(\sigma_1, (1, 1))$ act trivially on all of the given curves, the exponents $k_1'$ and $k_2'$ are irrelevant, and only the parity of $\ell$ is relevant). This completes the proof of the claim.

*Step 5.* Let $t = (\sigma_1^{-\ell}, (z^{-k_2'}, z^{-k_1'}))$. The braid $t$ lies in the centralizer of $\rho(B_n)$. Indeed, for $1 \leq i \leq n-1$ we have that $t\rho(\sigma_i)$ and $\rho(\sigma_i)t$ are both equal to

$$(1, (\sigma_i, \sigma_i)).$$

This calculation also shows that if we transvect $\rho$ by $t$ we obtain the diagonal inclusion map $B_n \to B_{2n}$. This completes the proof of the theorem.

□

## 12. Proof of Corollary 1.3

In this section we prove Corollary 1.3, which states that if $n \geq 7$ and $\rho : B_n' \to B_{2n-5}$ is a nontrivial homomorphism, then there is an automorphism $\tau$ of $B_{2n-5}$ so that $\tau \circ \rho$ is equal to the inclusion map. The argument given here is a simplification of an argument suggested by an anonymous referee for the paper by the second and third authors [28].

*Proof of Corollary 1.3.* There is a natural inclusion map $\nu : B_{n-2} \to B'_n$ given by $\nu(\sigma_i) = \sigma_i\sigma_{n-1}^{-1}$ for $i = 1, \ldots, n-3$. By Theorem 1.1, the composition

$$B_{n-2} \xrightarrow{\nu} B'_n \xrightarrow{\rho} B_{2n-5}$$

is equivalent to either the trivial map or the inclusion map. By post-composing with an automorphism of $B_{2n-5}$ we may assume that the composition is a transvection of either the trivial map or the inclusion map. We will show in the first case that $\rho$ is trivial and that in the second case $\rho$ is a transvection of the inclusion map.

Assume first that the composition is a transvection of the trivial map. Such a map has cyclic image, and so $\rho \circ \nu(\sigma_i)$ is independent of $i$. It follows that $\rho \circ \nu(\sigma_1\sigma_2^{-1}) = \rho(\sigma_1\sigma_2^{-1})$ is trivial. Since the normal closure of $\sigma_1\sigma_2^{-1}$ in $B'_n$ is $B'_n$ (Lemma 8.4), the map $\rho$ is trivial, as desired.

We now assume that the composition is a transvection of the inclusion map. This means that there is some $\beta$ in the centralizer of the image of $\rho \circ \nu$ so that $\rho \circ \nu(\sigma_i) = \sigma_i\beta$ for $i \in \{1, \ldots, n-3\}$. As in the previous paragraph, it follows that $\rho \circ \nu(\sigma_1\sigma_j^{-1}) = \rho(\sigma_1\sigma_j^{-1})$ is equal to $\sigma_1\sigma_j^{-1}$ for $j \in \{2, \ldots, n-3\}$. The group $B'_n$ is generated by $\sigma_1\sigma_j^{-1}$ for $j \in \{2, \ldots, n-1\}$ (see [32, p. 7] or [27, Prop. 3.1]), and so we focus our attention on the $\rho$-images of $\sigma_1\sigma_{n-2}^{-1}$ and $\sigma_1\sigma_{n-1}^{-1}$.

We claim that $\rho(\sigma_1\sigma_{n-2}^{-1})$ equals $\sigma_1 H_d^{-1}$ for some curve $d$ surrounding two marked points. We first observe that $\sigma_1\sigma_3^{-1}$ is conjugate to $\sigma_1\sigma_{n-2}^{-1}$ by an element $g \in B'_n$ that commutes with $\sigma_1^N z^{-1} \in B'_n$. It follows that $\rho(g)$ conjugates $\sigma_1\sigma_3^{-1}$ to $\rho(\sigma_1\sigma_{n-2}^{-1})$ and that $\rho(g)$ fixes the curve $c_1$. The claim follows.

We next claim that the curve $d$ has the following intersection numbers:

$$i(d, c_j) = \begin{cases} 0 & 1 \leq j \leq n-4 \\ 2 & j = n-3 \end{cases}$$

We first address the case $j = n-3$ . Since $\sigma_1\sigma_{n-3}^{-1}$ and $\sigma_1\sigma_{n-2}^{-1}$ satisfy the braid relation, the same is true for the images of these elements under $\rho$. From this it follows that $\sigma_{n-3}$ and $H_d$ satisfy the braid relation. As in Case 2 of the proof of Theorem 7.1, two half-twists satisfy the braid relation if and only if the corresponding curves intersect in exactly two points. Thus, $i(d, c_{n-3}) = 2$, as desired.

We now prove the claim in the case $1 \leq j \leq n-4$. Fix one such $j$. We have that $\sigma_j\sigma_{n-2}^{-1}$ is conjugate in $B'_n$ to $\sigma_1\sigma_3^{-1}$. The $\rho$-image of the latter is $\sigma_1\sigma_3^{-1}$ and so the $\rho$-image of $\sigma_j\sigma_{n-2}^{-1}$ must be a difference of commuting half-twists. We compute this image as follows:

$$\rho(\sigma_j\sigma_{n-2}^{-1}) = \rho(\sigma_j\sigma_1^{-1}\sigma_1\sigma_{n-2}^{-1}) = \rho(\sigma_j\sigma_1^{-1})\rho(\sigma_1\sigma_{n-2}^{-1}) = \sigma_j\sigma_1^{-1}\sigma_1 H_d^{-1} = \sigma_j H_d^{-1}.$$

By (a version of) the Thurston construction, the difference of two non-commuting half-twists is a partial pseudo-Anosov braid, and hence is not conjugate to $\sigma_1\sigma_3^{-1}$; see Step 3 of Case 4 of the proof of Theorem 1.1 in the paper by the second and third authors [28, Case 4, Step 3]. The claim follows.

By a similar argument to the previous claim, we have that $\rho(\sigma_1\sigma_{n-1}^{-1})$ is equal to a product $\sigma_1 H_e^{-1}$ where $H_e$ is a half-twist about a curve $e$ that surrounds two marked points and has the following intersection numbers:

$$i(e, c) = \begin{cases} 0 & c \in \{c_1, \ldots, c_{n-3}\} \\ 2 & c = d \end{cases}$$

By the change of coordinates principle for mapping class groups, there is a braid taking the chain of curves $(c_1, \ldots, c_{n-3}, d, e)$ to the standard chain $(c_1, \ldots, c_{n-1})$; let $\tau$ be the corresponding inner automorphism of $B_n$. The composition $\tau \circ \rho$ agrees with the inclusion map on the elements $\sigma_1\sigma_j$ with $2 \leq j \leq n-1$. Since these elements generate $B_n'$, the theorem is proven. □

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