POINT PUSHING, HOMOLOGY, AND THE HYPERELLIPTIC INVOLUTION

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ABSTRACT. The hyperelliptic Torelli group is the subgroup of the mapping class group consisting of elements that act trivially on the homology of the surface and that also commute with some fixed hyperelliptic involution. We establish a Birman exact sequence for hyperelliptic Torelli groups, and we show that this sequence splits. As a consequence, we show that the hyperelliptic Torelli group is generated by Dehn twists if and only if it is generated by reducible elements. We also give applications to the kernel of the integral Burau representation.

1. INTRODUCTION

Let S_g denote a closed, connected, orientable surface of genus g. The hyperelliptic Torelli group $\mathcal{SI}(S_g)$ is the subgroup of the mapping class group $\mathrm{Mod}(S_g)$ consisting of all elements that act trivially on $H_1(S_g;\mathbb{Z})$ and that commute with the isotopy class of some fixed hyperelliptic involution $s : S_g \to S_g$, that is, any order two homeomorphism acting by $-I$ on $H_1(S_g;\mathbb{Z})$. Every hyperelliptic involution of S_g is conjugate to the one shown in Figure 1.

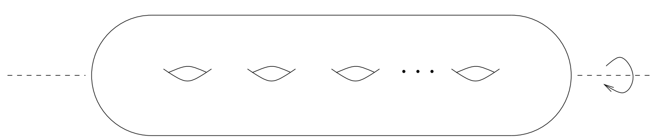

FIGURE 1. Rotation by π about the indicated axis is a hyperelliptic involution.

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The group $\mathcal{SI}(S_g)$ arises in algebraic geometry in the following context. Let $\mathcal{T}(S_g)$ denote the cover of the moduli space of Riemann surfaces corresponding to the Torelli subgroup $\mathcal{I}(S_g)$ of $\mathrm{Mod}(S_g)$. The period mapping is a function from $\mathcal{T}(S_g)$ to the Siegel upper half-space of rank g and is a 2-fold branched cover onto its image. The branch locus is the set of hyperelliptic points of $\mathcal{T}(S_g)$, the union of the fixed sets for the actions of the various hyperelliptic involutions on $\mathcal{T}(S_g)$. These fixed sets are pairwise disjoint, and the fundamental group of each component is isomorphic to $\mathcal{SI}(S_g)$. Because of this, $\mathcal{SI}(S_g)$ is related, for example, to the topological Schottky problem; see [9, Problem 1].

A basic tool in the theory of mapping class groups is the Birman exact sequence. This sequence relates the mapping class group of a surface with marked points to the mapping class group of the surface obtained by forgetting the marked points; see Section 3. This is a key ingredient for performing inductive arguments on the mapping class group. For instance, the standard proof that $\mathrm{Mod}(S_g)$ is generated by Dehn twists uses the Birman exact sequence in the inductive step on genus.

The main goal of this paper is to provide a Birman exact sequence for $\mathcal{SI}(S_g)$. As in the case of $\mathrm{Mod}(S_g)$, the Birman exact sequence is crucial for inductive arguments. As one such application, the authors and Childers showed that the top-dimensional homology of $\mathcal{SI}(S_g)$ is infinitely generated [6]. More recently, the authors and Putman used our Birman exact sequence to prove that $\mathcal{SI}(S_g)$ is generated by Dehn twists about s-invariant separating simple closed curves [7]; see the discussion after Theorem 1.3 below.

Hyperelliptic Torelli groups. In order to state our results, we need to define various hyperelliptic Torelli groups. First, the *mapping class group* $\mathrm{Mod}(S)$ of a surface S with set of marked points P is the group of homotopy classes of orientation-preserving homeomorphisms of S, where all homeomorphisms and homotopies are required to preserve P and fix ∂S pointwise. The *Torelli group* $\mathcal{I}(S)$ is then the subgroup of $\mathrm{Mod}(S)$ consisting of all elements that act trivially on the relative homology $H_1(S, P; \mathbb{Z})$.

A *hyperelliptic involution* of a surface S with marked points P is an order two homeomorphism of the pair (S, P) that acts by $-I$ on $H_1(S, P; \mathbb{Z})$.[1] Given a hyperelliptic involution of (S, P), we can define

[1]More generally, we could say that an involution s of a surface S with marked points P is hyperelliptic if it acts trivially on the homology of the space obtained

the *hyperelliptic mapping class group* $\mathrm{SMod}(S_g, P)$ as the group of isotopy classes of s-equivariant orientation-preserving homeomorphisms of the pair (S_g, P) that restrict to the identity on ∂S (isotopies are not required to be s-equivariant). The corresponding *hyperelliptic Torelli group* is

$$\mathcal{SI}(S_g, P) = \mathrm{SMod}(S_g, P) \cap \mathcal{I}(S_g, P).$$

In this paper, we will be interested in the hyperelliptic Torelli groups of a number of different surfaces of genus $g \geq 1$. We already described the surface S_g with its hyperelliptic involution s. We also need:

(1) $S_{g,1} = S_g$ with one s-invariant marked point p
(2) $S_{g,2} = S_g$ with a pair of marked points $P = \{p_1, p_2\}$ interchanged by s
(3) $S_g^1 = S_g$ minus the interior of a closed, embedded s-invariant disk.
(4) $S_g^2 = S_g$ minus the interiors of two closed, embedded, disjoint disks in S_g that are interchanged by s, plus an s-invariant pair of marked points $P = \{p_1, p_2\}$ in the resulting boundary[2]
(5) $S_{g,2}^1 = S_g^1$ plus a pair of interior marked points $P = \{p_1, p_2\}$ interchanged by s

In each of these cases s induces another hyperelliptic involution, which we also denote by s. Then the hyperelliptic Torelli group is defined as above. We emphasize that the above notations do not carry all of the important aspects of the definitions; for instance, it is important that the two marked points of $S_{g,2}$ are interchanged by s, etc.

Birman exact sequences. There are forgetful maps from the hyperelliptic Torelli groups of $S_{g,1}$, $S_{g,2}$, S_g^1, and S_g^2 to $\mathcal{SI}(S_g)$, and our Birman exact sequences give precise descriptions of the kernels. In the first case, we show that the kernel is trivial, and so the Birman exact sequence degenerates to an isomorphism.

Theorem 1.1. *Let $g \geq 1$. The forgetful map $\mathcal{SI}(S_{g,1}) \to \mathcal{SI}(S_g)$ is an isomorphism.*

from S by identifying pairs of points of P interchanged by s (cf. [18]); however, we will not require this level of generality.

[2]For symmetry, we could have defined S_g^1 so that it has an s-invariant pair of marked points on the boundary; however, the resulting hyperelliptic Torelli group would be equal to the one we defined.

Theorem 4.2 states that

$$\mathcal{SI}(S_g^1) \cong \mathcal{SI}(S_g) \times \mathbb{Z}.$$

On one hand, it is surprising to realize $\mathcal{SI}(S_g)$ as a subgroup of $\mathcal{SI}(S_g^1)$ since there is no embedding of S_g into S_g^1; on the other hand, in the course of the proof we will identify $\mathcal{SI}(S_g^1)$ with a subgroup of the pure braid group PB_{2g+1}, which is well known to split over its center.

Next we consider the kernel $\mathcal{SIBK}(S_{g,2})$ of the forgetful homomorphism $\mathcal{SI}(S_{g,2}) \to \mathcal{SI}(S_g)$ (the notation stands for "symmetric Torelli Birman kernel"). Theorem 3.2 below identifies the kernel of $\mathrm{SMod}(S_{g,2}) \to \mathrm{SMod}(S_g)$ with the free group F_{2g+1}, thought of as the fundamental group of a sphere with $2g+2$ punctures. So $\mathcal{SIBK}(S_{g,2})$ is identified with a subgroup of F_{2g+1}.

Denote the generators of F_{2g+1} by $\zeta_1, \ldots, \zeta_{2g+1}$ and the generators for $\mathbb{Z}^{2g+1}$ by $e_1, \ldots, e_{2g+1}$; the ζ_i are chosen so that they correspond to the loops shown in Figure 2. Denote by F_{2g+1}^{even} the subgroup of F_{2g+1} consisting of all even length words in the ζ_i. We will show in Section 4 that there is a homomorphism $\epsilon : F_{2g+1}^{even} \to \mathbb{Z}^{2g+1}$ defined by

$$\zeta_{i_1}^{\alpha_1} \zeta_{i_2}^{\alpha_2} \mapsto e_{i_1} - e_{i_2}$$

where $\alpha_j = \pm 1$ for each j. Since ϵ maps a nonabelian free group onto an infinite group, its kernel is an infinitely generated free group.

Theorem 1.2. *Let $g \geq 1$. If we identify $\mathcal{SIBK}(S_{g,2})$ with a subgroup of F_{2g+1} as above, then $\mathcal{SIBK}(S_{g,2}) = \ker \epsilon$ and the sequence*

$$1 \to \mathcal{SIBK}(S_{g,2}) \to \mathcal{SI}(S_{g,2}) \to \mathcal{SI}(S_g) \to 1$$

is split exact. In particular, $\mathcal{SI}(S_{g,2}) \cong \mathcal{SI}(S_g) \ltimes F_\infty$.

Again, the fact that the short exact sequence in Theorem 1.2 is split is unexpected because there is no embedding $S_g \to S_{g,2}$.

We can again ask about the case where the marked points are blown up to boundary components. It turns out that $\mathcal{SI}(S_g^2)$ is isomorphic to $\mathcal{SI}(S_{g,2})$ (see Section 4.2), so we can replace the latter with the former in Theorem 1.2. Our motivation for defining $\mathcal{SI}(S_g^2)$ is simply because this group is more naturally identified with a subgroup of PB_{2g+2}.

Mess [14] proved $\mathcal{SI}(S_2) \cong F_\infty$, and so the $g = 2$ cases of Theorems 4.2 and Theorem 1.2 give $\mathcal{SI}(S_2^1) \cong F_\infty \times \mathbb{Z}$ and $\mathcal{SI}(S_2^2) \cong F_\infty \ltimes F_\infty$.

Applications to generating sets. Hain has conjectured that in the case of a closed surface, the hyperelliptic Torelli group is generated by Dehn twists about separating curves that are preserved by the hyperelliptic involution [9, Conjecture 1] [15, Section 4]; such curves are called *symmetric*. As one step towards Hain's conjecture, we prove the following theorem. In the statement, an element of $\mathrm{Mod}(S_g)$ is *reducible* if it fixes a collection of isotopy classes of pairwise disjoint simple closed curves in S_g.

Theorem 1.3. *Let $g \geq 1$. Suppose that $\mathcal{SI}(S_k)$ is generated by Dehn twists about symmetric separating curves for $0 \leq k \leq g-1$. Then each reducible element of $\mathcal{SI}(S_g)$ is a product of Dehn twists about symmetric separating curves.*

By Theorem 1.3, Hain's conjecture is reduced to showing that the hyperelliptic Torelli group is generated by reducible elements. The isomorphism of Theorem 4.2 identifies reducible elements with reducible elements and identifies Dehn twists about symmetric separating curves with Dehn twists about symmetric separating curves. We can thus replace the $\mathcal{SI}(S_g)$ and $\mathcal{SI}(S_k)$ in Theorem 1.3 with $\mathcal{SI}(S_g^1)$ and $\mathcal{SI}(S_k^1)$.

The basic idea of the proof of Theorem 1.3 is to identify reducible elements of $\mathcal{SI}(S_g)$ with elements of $\mathcal{SI}(S_k^1)$ and $\mathcal{SI}(S_k^2)$ where $k < g$ and then to apply Theorem 4.2 and the following theorem.

Theorem 1.4. *For $g \geq 1$, each element of $\mathcal{SIBK}(S_{g,2})$ is a product of Dehn twists about symmetric separating simple closed curves. What is more, it suffices to use curves that either cut off a disk with two marked points or a genus 1 surface with two marked points.*

By work of Birman and Powell [4, 17], $\mathcal{I}(S_2)$ is generated by Dehn twists about separating curves. Moreover, every simple closed curve in S_2 is homotopic to a symmetric one (Fact 2.1). Thus Hain's conjecture is known to be true for $\mathcal{SI}(S_2)$; it follows that $\mathcal{SI}(S_2^1)$ and and $\mathcal{SI}(S_2^2)$ are also generated by Dehn twists about symmetric separating curves. As mentioned, the authors, together with Andrew Putman, have proven Hain's conjecture in general using Theorem 1.3.

We say that a separating curve in a surface has *genus* k if it cuts off a subsuface of genus k with one boundary component. In Section 6.4 we prove the following proposition.

Proposition 1.5. *Let $g \geq 1$. Every Dehn twist about a symmetric separating curve in S_g is equal to a product of Dehn twists about symmetric separating curves of genus 1 and 2.*

By Proposition 1.5, Hain's conjecture implies the stronger statement that $\mathcal{SI}(S_g)$ is generated by Dehn twists about symmetric separating curves of genus 1 and 2 only. We explain at the end of Section 6.4 how to derive the analogous result for $\mathcal{SI}(S_g^1)$ and $\mathcal{SI}(S_g^2)$.

Applications to the Burau representation. The (unreduced) Burau representation of the braid group B_n is a homomorphism $\mathrm{B}_n \to \mathrm{GL}(n, \mathbb{Z}[t, t^{-1}])$, and the integral Burau representation is the Burau representation evaluated at $t = -1$. When n is odd, the latter has a 1-dimensional trivial summand. We denote the kernel of the integral Burau representation by $\mathcal{BI}_n$ (for "braid Torelli").

The braid group B_n is isomorphic to the mapping class group of a disk D_n with n marked points. When $n = 2g+1$ there is a 2-fold cover of S_g^1 over D_{2g+1} branched over the marked points, and the nontrivial deck transformation is a hyperelliptic involution. Each element of B_{2g+1} lifts to an element of $\mathrm{SMod}(S_g^1)$, and the resulting composition $\mathrm{B}_{2g+1} \to \mathrm{SMod}(S_g^1) \to \mathrm{Sp}(2g, \mathbb{Z})$ is nothing other than the nontrivial summand of the integral Burau representation; see [16, Remark 4.3] and [13].

One version of the Birman–Hilden theorem [8, Theorem 9.2] tells us that the map $\mathrm{B}_{2g+1} \to \mathrm{SMod}(S_g^1)$ is an isomorphism, and so $\mathcal{SI}(S_g^1)$ is identified with $\mathcal{BI}_{2g+1}$. Similarly, we can identify $\mathcal{SI}(S_g^2) \cong \mathcal{SI}(S_{g,2})$ with $\mathcal{BI}_{2g+2}$.

By Theorem 4.2, $\mathcal{BI}_{2g+1}/Z(\mathcal{BI}_{2g+1}) \cong \mathcal{SI}(S_g)$. We can further use Theorem 1.2 to relate $\mathcal{BI}_{2g+2}$ to $\mathcal{BI}_{2g+1}$:

$$\mathcal{BI}_{2g+2} \cong (\mathcal{BI}_{2g+1}/Z(\mathcal{BI}_{2g+1})) \ltimes F_\infty.$$

Our methods do not give an analogous decomposition of $\mathcal{BI}_{2g+1}$. Indeed, there is not even a natural homomorphism $\mathcal{BI}_{2g+1} \to \mathcal{BI}_{2g} = \mathcal{BI}_{2g}/Z(\mathcal{BI}_{2g})$.

Under the isomorphisms $\mathcal{SI}(S_g^1) \to \mathcal{BI}_{2g+1}$ and $\mathcal{SI}(S_g^2) \to \mathcal{BI}_{2g+2}$, Dehn twists about symmetric separating curves correspond to squares of Dehn twists about curves surrounding odd numbers of marked points. Thus Hain's conjecture can be translated as: the group $\mathcal{BI}_n$ is generated by Dehn twists about curves surrounding odd numbers. It is classically known that $\mathcal{BI}_3 \cong \mathbb{Z}$, and Smythe proved that $\mathcal{BI}_4$ is generated by squares of Dehn twists about curves surrounding 3 marked points [19] (see also Proposition 5.3 below). Since $\mathcal{SI}(S_2^1)$, and $\mathcal{SI}(S_2^2)$ are both generated by Dehn twists about symmetric separating curves, we have the following consequence.

Corollary 1.6. *For $n \leq 6$, the group $\mathcal{BI}_n$ is generated by squares of Dehn twists about curves surrounding odd numbers of marked points.*

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2. The Birman–Hilden theorem

In this section, we recall some special cases of a theorem of Birman and Hilden. In this section, and throughout the paper, we denote by σ the homotopy class of the hyperelliptic involution s.

The Birman–Hilden theorem for closed surfaces. For $g \leq 2$, the group $\mathrm{Mod}(S_g)$ has a generating set consisting of Dehn twists about symmetric simple closed curves. Each such Dehn twist has a representative that commutes with s, and so we obtain the following.

Fact 2.1. *For $g \leq 2$, we have $\mathrm{SMod}(S_g) = \mathrm{Mod}(S_g)$.*

For $g \geq 3$, the group $\mathrm{SMod}(S_g)$ has infinite index in $\mathrm{Mod}(S_g)$. Indeed, if a is any isotopy class of simple closed curves in S_g that is not fixed by σ, then no nontrivial power of the Dehn twist T_a is an element of $\mathrm{SMod}(S_g)$ (note that, by Fact 2.1 no such curves exist in a closed genus two surface!). However, there is a very useful description of $\mathrm{SMod}(S_g)$ given by Birman–Hilden, which we now explain.

The quotient of S_g by s is a sphere $S_{0,2g+2}$ with $2g+2$ marked points, namely the images of the fixed points of s. By definition, any element f of $\mathrm{SMod}(S_g)$ has an s-equivariant representative ϕ, and hence descends to a homeomorphism $\overline{\phi}$ of $S_{0,2g+2}$. Birman and Hilden [5] proved that if ϕ is isotopic to the identity, then $\overline{\phi}$ is isotopic to the identity (equivalently, ϕ is s-equivariantly isotopic to the identity). In other words, there is a well-defined homomorphism $\theta : \mathrm{SMod}(S_g) \to$

$\mathrm{Mod}(S_{0,2g+2})$. This map is surjective, as the standard half-twist generators for $\mathrm{Mod}(S_{0,2g+2})$ all lift to homeomorphisms of S_g. We summarize this discussion in the following theorem.

Theorem 2.2 (Birman–Hilden)**.** *For $g \geq 2$, there is a short exact sequence:*

$$1 \to \langle \sigma \rangle \to \mathrm{SMod}(S_g) \xrightarrow{\theta} \mathrm{Mod}(S_{0,2g+2}) \to 1.$$

A Birman–Hilden theorem for surfaces with marked points. The quotient $S_{g,2}/\langle s \rangle$ is the pair $(S_{0,2g+2}, \overline{p})$, where $\overline{p} \in S_{0,2g+2}$ is the image of the pair of marked points in $S_{g,2}$. Elements of $\mathrm{Mod}(S_{0,2g+2}, \overline{p})$ can permute the $2g+2$ marked points coming from $S_{0,2g+2}$, but must preserve the marked point $\overline{p}$. We have the following analogue of Theorem 2.2.

Theorem 2.3 (Birman–Hilden)**.** *For $g \geq 1$, there is a short exact sequence:*

$$1 \to \langle \sigma \rangle \to \mathrm{SMod}(S_{g,2}) \xrightarrow{\theta} \mathrm{Mod}(S_{0,2g+2}, \overline{p}) \to 1.$$

The Birman–Hilden theorem for the torus. Theorem 2.2 does not hold as stated for $g = 1$. In fact, in this case, the map θ is not even well defined. This is because there are nontrivial finite order homeomorphisms of $S_{0,4}$ that lift to homeomorphisms of S_1 that are homotopic to the identity. Therefore, we are forced to redefine θ in this case. We have

$$\mathrm{SMod}(S_{1,1}) = \mathrm{Mod}(S_{1,1}) \cong \mathrm{Mod}(S_1) = \mathrm{SMod}(S_1) \cong \mathrm{SL}(2, \mathbb{Z}).$$

Let $\overline{p}$ denote the image of the marked point of $S_{1,1}$ in $S_{0,4}$ (this is already one of the 4 marked points). For $g = 1$, we can then define θ via the composition

$$\mathrm{SMod}(S_1) \xrightarrow{\cong} \mathrm{SMod}(S_{1,1}) \to \mathrm{Mod}(S_{0,4}, \overline{p}),$$

where $\mathrm{Mod}(S_{0,4}, \overline{p})$ is the subgroup of $\mathrm{Mod}(S_{0,4})$ consisting of elements that fix the marked point $\overline{p}$. We then have the following genus 1 version of Theorem 2.2.

Theorem 2.4 (Birman–Hilden)**.** *There is a short exact sequence:*

$$1 \to \langle \sigma \rangle \to \mathrm{SMod}(S_1) \xrightarrow{\theta} \mathrm{Mod}(S_{0,4}, \overline{p}) \to 1.$$

3. Birman exact sequences for the hyperelliptic mapping class group

In this section we give Birman exact sequences for hyperelliptic mapping class groups in the two cases which will be of interest for us: first, forgetting one marked point, and then forgetting two.

We begin by recalling the classical Birman exact sequence. Let S denote a connected, orientable, compact surface with finitely many marked points in its interior. Assume that the surface S° obtained by removing the marked points from S has negative Euler characteristic. Let $p \in S$ be an additional marked point (distinct from any others in S). There is a forgetful map $\mathrm{Mod}(S,p) \to \mathrm{Mod}(S)$, and the Birman exact sequence identifies the kernel of this map with $\pi_1(S^\circ, p)$:

$$1 \to \pi_1(S^\circ, p) \stackrel{\mathcal{P}ush}{\to} \mathrm{Mod}(S,p) \to \mathrm{Mod}(S) \to 1.$$

Given an element α of $\pi_1(S^\circ, p)$, we can describe $\mathcal{P}ush(\alpha)$ as the map obtained by pushing p along α; see [8, Section 5.2] or [3, Section 1].

3.1. Forgetting one point.

As in the classical Birman exact sequence, there is a forgetful map $\mathrm{SMod}(S_{g,1}) \to \mathrm{SMod}(S_g)$.

Theorem 3.1. *Let $g \geq 1$. The forgetful map* $\mathrm{SMod}(S_{g,1}) \to \mathrm{SMod}(S_g)$ *is injective.*

Note that this map is not surjective for $g \geq 2$. For example, its image does not contain a Dehn twist about a symmetric curve through the marked point.

Proof. We already said $\mathrm{SMod}(S_{1,1}) \cong \mathrm{SMod}(S_1)$. So assume $g \geq 2$. The classical Birman exact sequence for a surface of genus $g \geq 2$ is:

$$1 \to \pi_1(S_g) \stackrel{\mathcal{P}ush}{\to} \mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(S_g) \to 1.$$

Therefore, to prove the theorem, we need to show that the image of $\pi_1(S_g) = \pi_1(S_g, p)$ in $\mathrm{Mod}(S_{g,1})$ intersects $\mathrm{SMod}(S_{g,1})$ trivially (as usual, p is the marked point of $S_{g,1}$). In other words, we need to show that $\sigma \in \mathrm{SMod}(S_{g,1})$ does not commute with any nontrivial element of the image of $\pi_1(S_g)$. For $f \in \mathrm{Mod}(S_{g,1})$ and $\alpha \in \pi_1(S_g)$, we have that $f\mathcal{P}ush(\alpha)f^{-1} = \mathcal{P}ush(f_\star(\alpha))$. Therefore, we need to show that $\sigma_\star(\alpha) \neq \alpha$ for all nontrivial $\alpha \in \pi_1(S_g)$.

Choose a hyperbolic metric on S_g so that s is an isometry. A concrete way to do this is to identify S_g with a regular hyperbolic $(4g+2)$-gon with opposite sides glued, and take s to be rotation by π through the center.

Next, choose a universal covering $\mathbb{H}^2 \to S_g$. The preimage of p in $\mathbb{H}^2$ is the set $\{\gamma \cdot \widetilde{p} : \gamma \in \pi_1(S_g)\}$, where $\widetilde{p}$ is some fixed lift of p.

The map s has a unique lift $\widetilde{s}$ to $\mathrm{Isom}^+(\mathbb{H}^2)$ that fixes $\widetilde{p}$. This lift has order two. By the classification of elements of $\mathrm{Isom}^+(\mathbb{H}^2)$, it is a rotation by π. Thus, $\widetilde{s}$ has exactly one fixed point.

The action of $\widetilde{s}$ on the set $\{\gamma \cdot \widetilde{p}\}$ is given by

$$\gamma \cdot \widetilde{p} \mapsto \sigma_\star(\gamma) \cdot \widetilde{p}.$$

If $\sigma_\star(\alpha) = \alpha$, then it follows that $\widetilde{s}$ fixes $\alpha \cdot \widetilde{p}$. But we already said that $\widetilde{s}$ has a unique fixed point, namely $\widetilde{p}$. So $\alpha = 1$, as desired. $\square$

3.2. **Forgetting two points.** Let $\overline{p}$ denote the image in $S_{0,2g+2}$ of the pair of marked points of $S_{g,2}$. Let $\mathcal{SBK}(S_{g,2})$ denote the kernel of the forgetful homomorphism $\mathrm{SMod}(S_{g,2}) \to \mathrm{SMod}(S_g)$ (the notation is for "symmetric Birman kernel"). We have a short exact sequence:

$$1 \to \mathcal{SBK}(S_{g,2}) \to \mathrm{SMod}(S_{g,2}) \to \mathrm{SMod}(S_g) \to 1.$$

Here and throughout, when we write $\pi_1(S_{0,2g+2})$, we mean the fundamental group of the punctured sphere obtained by removing the $2g+2$ marked points from $S_{0,2g+2}$.

Theorem 3.2. *Let $g \geq 1$. We have that $\mathcal{SBK}(S_{g,2}) \cong F_{2g+1}$, where F_{2g+1} is identified with $\pi_1(S_{0,2g+2}, \overline{p})$.*

Proof. We have the following commutative diagram.

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & \langle\sigma\rangle \arrow[r, "\cong"] \arrow[d] & \langle\sigma\rangle \arrow[d] & \\
1 \arrow[r] & \mathcal{SBK}(S_{g,2}) \arrow[r] & \mathrm{SMod}(S_{g,2}) \arrow[r] \arrow[d] & \mathrm{SMod}(S_g) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(S_{0,2g+2},\overline{p}) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{Mod}(S_{0,2g+2},\overline{p}) \arrow[r] \arrow[d] & \mathrm{Mod}(S_{0,2g+2}) \arrow[r] & 1 \\
 & F_{2g+1} & 1 & &
\end{tikzcd}$$

The second horizontal short exact sequence is an instance of the Birman exact sequence, and the two vertical sequences are given by Theorems 2.2, 2.3, and 2.4. From the diagram it is straightforward to see that $\mathcal{SBK}(S_{g,2}) \cong \pi_1(S_{0,2g+2})$. □

4. Birman exact sequences for hyperelliptic Torelli groups

The main results of this paper are Birman exact sequences for hyperelliptic Torelli groups. As in the previous section, there are two versions, corresponding to forgetting one point (Theorem 1.1) and forgetting two points (Theorem 1.2).

4.1. **Forgetting one point.** Let $\mathrm{PMod}(S_{0,2g+2})$ denote the subgroup of $\mathrm{Mod}(S_{0,2g+2})$ consisting of elements that induce the trivial permutation of the marked points. The next fact follows from the discussion after Lemma 1 of [1].

Lemma 4.1. *Let $g \geq 1$. Under the map $\theta : \mathrm{SMod}(S_g) \to \mathrm{Mod}(S_{0,2g+2})$, the image of $\mathcal{SI}(S_g)$ lies in* $\mathrm{PMod}(S_{0,2g+2})$.

We are now ready for the proof of our first Birman exact sequence for hyperelliptic Torelli groups.

Proof of Theorem 1.1. Since $H_1(S_g, p; \mathbb{Z})$ and $H_1(S_g; \mathbb{Z})$ are canonically isomorphic, the forgetful map $\mathrm{SMod}(S_{g,1}) \to \mathrm{SMod}(S_g)$ restricts to a homomorphism $\mathcal{SI}(S_{g,1}) \to \mathcal{SI}(S_g)$, and it follows immediately from Theorem 3.1 that this map is injective. We will show that it is also surjective. Let $f \in \mathcal{SI}(S_g)$ and let ϕ be a representative homeomorphism that commutes with s. By Lemma 4.1, the induced homeomorphism of $S_{0,2g+2}$ fixes each of the $2g+2$ marked points. It follows that ϕ fixes the marked point $p \in S_{g,1}$ and that ϕ represents an element $\widetilde{f}$ of $\mathcal{SI}(S_{g,1})$ that maps to f. □

Before moving on to the second Birman exact sequence for the hyperelliptic Torelli group, we give a variation of Theorem 1.1, where we forget a boundary component (really, cap a boundary component) instead of a marked point.

Capping a boundary component. The inclusions $S_g^1 \to S_g$ and $S_{g,2}^1 \to S_{g,2}$ induce homomorphisms on the level of hyperelliptic Torelli groups, and so we can again ask about the kernel.

Theorem 4.2. *Let $g \geq 1$. The inclusions $S_g^1 \to S_g$ and $S_{g,2}^1 \to S_{g,2}$ induce isomorphisms*

$$\mathcal{SI}(S_g^1) \cong \mathcal{SI}(S_g) \times \mathbb{Z}$$
$$\mathcal{SI}(S_{g,2}^1) \cong \mathcal{SI}(S_{g,2}) \times \mathbb{Z}.$$

In both cases the $\mathbb{Z}$ factor is the Dehn twist about the boundary.

Proof. We treat the case of S_g^1, with the case of $S_{g,2}^1$ being essentially the same.

As in the introduction, $\mathrm{SMod}(S_g^1)$ is isomorphic to B_{2g+1}, which we identify with $\mathrm{Mod}(D_{2g+1})$. By (a version of) Lemma 4.1, the group $\mathcal{SI}(S_g^1)$ is identified with a subgroup of PB_{2g+1}.

For any n, the group PB_n splits as a direct product over its center, which is generated by the Dehn twist $T_{\partial D_n}$ [8, Section 9.3]. Under the restriction $\bar{\theta} : \mathcal{SI}(S_g^1) \hookrightarrow \mathrm{PB}_{2g+1}$, we have $\bar{\theta}^{-1}(Z(\mathrm{PB}_{2g+1})) = \langle T_{\partial S_g^1} \rangle$. Thus, $\mathcal{SI}(S_g^1)$ splits as a direct product over $\langle T_{\partial S_g^1} \rangle$.

It remains to show that $\mathcal{SI}(S_g^1)/\langle T_{\partial S_g^1} \rangle \cong \mathcal{SI}(S_g)$. There is a short exact sequence

$$1 \to \langle T_{\partial S_g^1} \rangle \to \mathrm{Mod}(S_g^1) \to \mathrm{Mod}(S_{g,1}) \to 1,$$

where the map $\mathrm{Mod}(S_g^1) \to \mathrm{Mod}(S_{g,1})$ is the one induced by the inclusion $S_g^1 \to S_{g,1}$; see [8, Proposition 3.19]. On the level of hyperelliptic

Torelli groups, this gives

$$1 \to \langle T_{\partial S_g^1} \rangle \to \mathcal{SI}(S_g^1) \to \mathcal{SI}(S_{g,1}) \to 1.$$

We have already shown that $\mathcal{SI}(S_{g,1}) \cong \mathcal{SI}(S_g)$ (Theorem 1.1). Thus, $\mathcal{SI}(S_g^1)/\langle T_{\partial S_g^1} \rangle \cong \mathcal{SI}(S_g)$, and we are done. $\square$

4.2. **Forgetting two points.** Recall from Theorem 3.2 that the kernel of $\mathrm{SMod}(S_{g,2}) \to \mathrm{SMod}(S_g)$ is $\mathcal{SBK}(S_{g,2}) \cong F_{2g+1}$, which is identified with $\pi_1(S_{0,2g+2}, \overline{p})$, where $\overline{p}$ is the image in $S_{0,2g+2}$ of the pair P of marked points of $S_{g,2}$. Let $\zeta_1, \ldots, \zeta_{2g+1}$ be the generators for $\pi_1(S_{0,2g+2}, \overline{p}) \cong F_{2g+1}$ shown in Figure 2. In what follows, we identify F_{2g+1} with $\pi_1(S_{0,2g+2}, \overline{p}) = \langle \zeta_1, \ldots, \zeta_{2g+1} \rangle$.

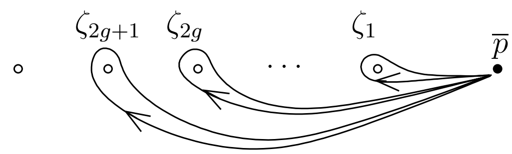

FIGURE 2. The elements ζ_i of $\pi_1(S_{0,2g+2}, p)$.

Since a homology between two simple closed curves in $S_{g,2}$ descends to a homology between the corresponding curves in S_g, the forgetful map $\mathrm{SMod}(S_{g,2}) \to \mathrm{SMod}(S_g)$ restricts to a homomorphism $\mathcal{SI}(S_{g,2}) \to \mathcal{SI}(S_g)$. We now set about describing the kernel.

As in the introduction, the *even subgroup* of F_{2g+1} is the kernel of the homomorphism $F_{2g+1} \to \mathbb{Z}/2\mathbb{Z}$ sending each ζ_i to 1. The elements of F_{2g+1}^{even} are products $\zeta_{i_1}^{\alpha_1} \zeta_{i_2}^{\alpha_2} \cdots \zeta_{i_k}^{\alpha_k}$ where k is even and $\alpha_i \in \{-1, 1\}$.

Let

$$\epsilon : F_{2g+1}^{even} \to \mathbb{Z}^{2g+1}$$

be the map given by

$$\zeta_{i_1}^{\alpha_1} \zeta_{i_2}^{\alpha_2} \cdots \zeta_{i_k}^{\alpha_k} \mapsto \sum_{j=1}^{k} (-1)^{j+1} e_{i_j}$$

where $\{e_1, \ldots, e_{2g+1}\}$ are the standard generators for $\mathbb{Z}^{2g+1}$.

Lemma 4.3. *The map ϵ is a well-defined homomorphism.*

Proof. It follows from the Reidemeister–Schreier method that F_{2g+1}^{even} is freely generated by elements of the form ζ_1^2 and $\zeta_1 \zeta_i^{\pm 1}$ with $i > 1$.

Therefore, there is a well-defined homomorphism $\epsilon_0 : F_{2g+1}^{even} \to \mathbb{Z}^{2g+1}$ given by

$$\zeta_1\zeta_i^{\pm 1} \mapsto e_1 - e_i.$$

Given any other $\zeta_{i_1}^{\alpha_1}\zeta_{i_2}^{\alpha_2}$, we can rewrite it as a product of at most two generators, and we can check that in all cases $\epsilon_0(\zeta_{i_1}^{\alpha_1}\zeta_{i_2}^{\alpha_2}) = \epsilon(\zeta_{i_1}^{\alpha_1}\zeta_{i_2}^{\alpha_2}) = e_{i_1} - e_{i_2}$. Likewise, it follows that the homomorphism ϵ_0 agrees with ϵ on any element $\zeta_{i_1}^{\alpha_1}\zeta_{i_2}^{\alpha_2}\cdots\zeta_{i_k}^{\alpha_k}$, and we are done. □

Theorem 1.2 states that, as a subgroup of $\mathcal{SBK}(S_{g,2})$, the group $\mathcal{SIBK}(S_{g,2})$ is equal to the image of $\ker \epsilon$ under the isomorphism

$$\langle \zeta_1, \ldots, \zeta_{2g+1} \rangle = F_{2g+1} \cong \pi_1(S_{0,2g+2}, \overline{p}) \xrightarrow{\cong} \mathcal{SBK}(S_{g,2}).$$

In order to prove Theorem 1.2, we will need two lemmas describing the action of elements of $\mathcal{SBK}(S_{g,2})$ on the relative homology $H_1(S_g, P; \mathbb{Z})$ (throughout this section, P is the pair of marked points of $S_{g,2}$). Our argument has its origins in the work of Johnson [11, Section 2], van den Berg [20, Section 2.4], and Putman [18, Section 4].

A *proper arc* α in a surface S with marked points $\{p_i\}$ is a map $\alpha : [0,1] \to (S, \{p_i\})$ where $\alpha^{-1}(\{p_i\}) = \{0, 1\}$.

Lemma 4.4. *Let $g \geq 1$. If f is an element of $\mathcal{SBK}(S_{g,2})$, and if β is any oriented proper arc in $(S_{g,2})$ connecting the two marked points, then f is an element of $\mathcal{SIBK}(S_{g,2})$ if and only if in $H_1(S_g, P; \mathbb{Z})$ we have $f_\star([\beta]) = [\beta]$.*

Proof. One direction is trivial: if $f \in \mathcal{SIBK}(S_{g,2})$, then by definition, f acts trivially on $H_1(S_g, P; \mathbb{Z})$.

We now prove the other direction. There is a basis for $H_1(S_g, P; \mathbb{Z})$ given by (the classes of) finitely many oriented closed curves plus the oriented arc β. Thus, to prove the lemma, we only need to show that any $f \in \mathcal{SBK}(S_{g,2})$ preserves the class in $H_1(S_g, P; \mathbb{Z})$ of each oriented closed curve in S_g.

Let ϕ be a representative of f. We can regard ϕ either as a homeomorphism of $S_{g,2}$ or as a homeomorphism of S_g. Also, let γ be an oriented closed curve in S_g. We can similarly regard γ as a representative of an element of either $H_1(S_g; \mathbb{Z})$ or of $H_1(S_g, P; \mathbb{Z})$.

Since $f \in \mathcal{SBK}(S_{g,2})$, it follows that ϕ is isotopic to the identity as a homeomorphism of S_g. In particular, we have

$$[\gamma] = [\phi(\gamma)] \in H_1(S_g; \mathbb{Z}).$$

There is a natural map $H_1(S_g; \mathbb{Z}) \to H_1(S_g, P; \mathbb{Z})$ where $[\gamma]$ maps to $[\gamma]$ and $[\phi(\gamma)]$ maps to $[\phi(\gamma)]$. Since this map is well defined, it follows that

$$[\gamma] = [\phi(\gamma)] \in H_1(S_g, P; \mathbb{Z}),$$

which is what we wanted to show. □

Lemma 4.4 tell us that in order to show that an element of the group $\mathcal{SBK}(S_{g,2})$ lies in the Torelli group, we only need to keep track of its action on the homology class of a single arc. The only other ingredient we need in order to prove Theorem 1.2 is a formula for how elements of $\mathcal{SBK}(S_{g,2})$ act on such classes.

Via the isomorphism $\pi_1(S_{0,2g+2}, \overline{p}) \to \mathcal{SBK}(S_{g,2})$, there is an action of $\pi_1(S_{0,2g+2}, \overline{p})$ on $H_1(S_g, P; \mathbb{Z})$. We denote the action of $\zeta \in \pi_1(S_{0,2g+2}, \overline{p})$ by $\zeta_\star$.

Each generator ζ_i of $\pi_1(S_{0,2g+2}, \overline{p})$ is represented by a simple loop in $S_{0,2g+2}$ based at $\overline{p}$ (see Figure 2). The loop associated to ζ_i lies in the regular neighborhood of an arc in $S_{0,2g+2}$ that connects $\overline{p}$ to the ith marked point of $S_{0,2g+2}$. We denote the preimage in $S_{g,2}$ of the ith such arc in $(S_{0,2g+2}, \overline{p})$ by β_i. We orient the β_i so that they all emanate from the same marked point; see Figure 3.

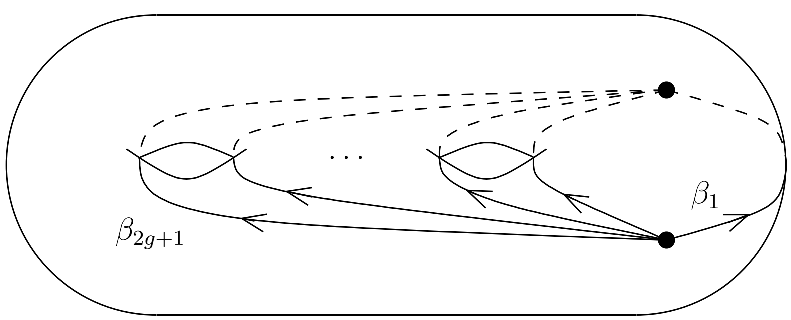

FIGURE 3. The arcs β_i in $S_{g,2}$.

For each i, we choose a neighborhood N_i of β_i that is fixed by s. A *half-twist* about β_i is a homeomorphism of $S_{g,2}$ that is the identity on the complement of N_i and is described on N_i by Figure 4. This half-twist is well defined as a mapping class.

Lemma 4.5. *Let $g \geq 1$, let $\zeta \in \pi_1(S_{0,2g+2}, \overline{p})$, and say*
$$\zeta = \zeta_{i_1}^{\alpha_1} \cdots \zeta_{i_m}^{\alpha_m}$$
where $\zeta_{i_j} \in \{\zeta_i\}$ and $\alpha_i \in \{-1, 1\}$. We have the following formula for the action on $H_1(S_g, P; \mathbb{Z})$:
$$\zeta_\star([\beta_k]) = [\beta_k] + 2\sum_{j=1}^{m}(-1)^j[\beta_{i_j}].$$

Proof. First of all, we claim that the image of ζ_i under the isomorphism $\pi_1(S_{0,2g+2}, \overline{p}) \to \mathcal{SBK}(S_{g,2})$ is the half-twist about β_i. Indeed, the image of ζ_i under the map $\pi_1(S_{0,2g+2}, \overline{p}) \to \mathrm{Mod}(S_{0,2g+2}, \overline{p})$ is a Dehn twist about the boundary of a regular neighborhood of ζ_i, and the unique lift of this Dehn twist to $\mathcal{SBK}(S_{g,2})$ is a half-twist about β_i.

We can now determine the action of $\zeta \in \pi_1(S_{0,2g+2}, \overline{p})$ on $[\beta_k] \in H_1(S_g, P; \mathbb{Z})$. We first deal with the case where $\zeta = \zeta_i^{\pm 1}$. If $i = k$, then we immediately see that the half-twist about β_i (or its inverse) simply reverses the orientation of β_k, and so we have
$$\zeta_\star([\beta_k]) = -[\beta_k] = [\beta_k] - 2[\beta_k] = [\beta_k] - 2[\beta_i],$$
and the lemma is verified in this case.

If $\zeta = \zeta_i$ where $\zeta_i \neq \zeta_k$, then a neighborhood of $\beta_i \cup \beta_k$ in $S_{g,2}$ is an annulus with two marked points. As above, $\zeta = \zeta_i$ maps to the half-twist about β_i. Simply by drawing the picture of the action (see Figure 4), we check the formula:
$$\zeta_\star([\beta_k]) = [\beta_k] - 2[\beta_i].$$
The case $\zeta = \zeta_i^{-1}$ is similar.

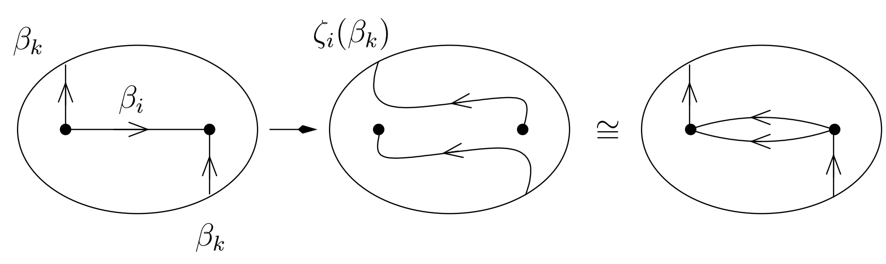

Figure 4. The action of the half-twist about β_i on β_k.

Since the action of $\mathcal{SBK}(S_{g,2})$ on $H_1(S_g, P; \mathbb{Z})$ is linear, we can now complete the proof of the lemma by induction. Suppose the lemma holds for $m-1$, that is, the induced action of $\zeta_{i_1}^{\alpha_i}\cdots\zeta_{i_{m-1}}^{\alpha_{m-1}}$ on $[\beta_k]$ is

$$[\beta_k] \mapsto [\beta_k] + 2\sum_{j=1}^{m-1}(-1)^j[\beta_{i_j}].$$

By linearity, and applying the case where $\zeta = \zeta_i^{\pm 1}$, the image of the latter homology class under $\zeta_{i_m}^{\alpha_m}$ is

$$([\beta_k] - 2[\beta_{i_m}]) + 2\sum_{j=1}^{m-1}(-1)^j\left([\beta_{i_j}] - 2[\beta_{i_m}]\right),$$

which we rewrite as

$$[\beta_k] + \left(2\sum_{j=1}^{m-1}(-1)^j[\beta_{i_j}]\right) + \left(4\sum_{j=1}^{m-1}(-1)^{j+1}[\beta_{i_m}]\right) - 2[\beta_{i_m}].$$

The sum of the third and fourth terms is $2(-1)^j[\beta_{i_m}]$, and so the lemma is proven. $\square$

We are now poised to prove Theorem 1.2, which states that the map $\mathcal{SI}(S_{g,2}) \to \mathcal{SI}(S_g)$ is surjective and identifies its kernel with $\ker\epsilon$.

Proof of Theorem 1.2. By Lemma 4.4, an element of $\mathcal{SBK}(S_{g,2})$ lies in $\mathcal{SIBK}(S_{g,2})$ if and only if it fixes the relative class $[\beta_1]$ in $H_1(S_g, P; \mathbb{Z})$. It then follows from Lemma 4.5 that an element of $\mathcal{SBK}(S_{g,2})$ fixes $[\beta_1]$ if and only if it lies in the image of $\ker\epsilon$.

It remains to show that there is a splitting $\mathcal{SI}(S_g) \to \mathcal{SI}(S_{g,2})$. By Theorem 4.2, there is an injective homomorphism $\mathcal{SI}(S_g) \to \mathcal{SI}(S_g^1)$ with a left inverse induced by the inclusion $S_g^1 \to S_g$. Via the (symmetric) inclusion $S_g^1 \to S_{g,2}$, we obtain an injective homomorphism $\mathcal{SI}(S_g^1) \to \mathcal{SI}(S_{g,2})$. The composition is the desired map $\mathcal{SI}(S_g) \to \mathcal{SI}(S_{g,2})$. $\square$

Capping two boundary components. We now aim to give an analogue of Theorem 4.2, that is, we would like to give a version of Theorem 1.2 for a surface with two boundary components instead of two punctures. The kernel of the map $\mathrm{SMod}(S_g^2) \to \mathrm{SMod}(S_{g,2})$ obtained by capping both boundary components with marked disks is $\langle T_{\partial_1 S_g^2} T_{\partial_2 S_g^2}\rangle \cong \mathbb{Z}$, where $\partial_1 S_g^2$ and $\partial_2 S_g^2$ are the two boundary components of S_g^2; see Lemma 6.3 below. This element does not act trivially on $H_1(S_g^2, P; \mathbb{Z})$ (here P is the pair of marked points in ∂S_g^2), and so we conclude that the forgetful map $\mathcal{SI}(S_g^2) \to \mathcal{SI}(S_{g,2})$ is an isomorphism. Therefore, we obtain the boundary-capping version of Theorem 1.2 by simply replacing $\mathcal{SI}(S_{g,2})$ with $\mathcal{SI}(S_g^2)$.

5. Generating $\mathcal{SIBK}(S_{g,2})$ by products of twists

We will now use Theorem 1.2 in order to prove Theorem 1.4, which states that $\mathcal{SIBK}(S_{g,2})$ is generated by Dehn twists about symmetric separating curves that either cut off a disk with two marked points or a genus 1 surface with two marked points.

Theorem 1.2 identifies the group $\mathcal{SIBK}(S_{g,2})$ with $\ker \epsilon$, and so that is where we begin. It is a general fact from combinatorial group theory that the kernel of a homomorphism is normally generated by elements that map to the defining relators for the image of the homomorphism. We aim to exploit this fact, and so we start by determining the image of ϵ.

Let $\mathbb{Z}^{2g+1} \to \mathbb{Z}$ be the map that records the sum of the coordinates, and let $\mathbb{Z}^{2g+1}_{bal}$ be the kernel.

Lemma 5.1. *Let $g \geq 0$. The image of F^{even}_{2g+1} under ϵ is $\mathbb{Z}^{2g+1}_{bal}$.*

Proof. It follows immediately from the definition of the map ϵ that $\epsilon(F^{even}_{2g+1})$ lies in $\mathbb{Z}^{2g+1}_{bal}$. To show that $\epsilon(F^{even}_{2g+1})$ is all of $\mathbb{Z}^{2g+1}_{bal}$, it suffices to show that $\mathbb{Z}^{2g+1}_{bal}$ is generated by the elements $\epsilon(\zeta_i\zeta_j) = e_i - e_j$, where e_i is a generator the ith factor of $\mathbb{Z}^{2g+1}$.

Let $\mathbb{Z}^{2g+1} \to \mathbb{Z}$ be the function that records the sum of the absolute values of the coordinates. We think of this function as a height function. The only element of $\mathbb{Z}^{2g+1}$ at height zero is the identity, which is the image of the identity element of F^{even}_{2g+1}. Let z be an arbitrary nontrivial element of $\mathbb{Z}^{2g+1}_{bal}$. Since z is nontrivial, it has at least one nonzero component, say the mth. By the definition of $\mathbb{Z}^{2g+1}_{bal}$, there must be one component, say the jth, with opposite sign. Say the mth component is negative and the jth component is positive. The sum $\epsilon(\zeta_i\zeta_j) + z$ has height strictly smaller than that of z, so by induction the lemma is proven. □

Lemma 5.2. *Let $g \geq 0$. The group $\mathbb{Z}^{2g+1}_{bal}$ has a presentation:*

$$\langle e_{1,1}, \ldots, e_{2g+1,2g+1}, e_{2,1}, \ldots, e_{2g+1,1} \mid e_{i,i} = 1, [e_{i,1}, e_{j,1}] = 1 \rangle.$$

Proof. Since $\mathbb{Z}^{2g+1}_{bal}$ is the subgroup of $\mathbb{Z}^{2g+1}$ described by one linear equation (the sum of the coordinates is 0), we see that $\mathbb{Z}^{2g+1}_{bal} \cong \mathbb{Z}^{2g}$. Denote by η the isomorphism given $\mathbb{Z}^{2g+1}_{bal} \to \mathbb{Z}^{2g}$ given by forgetting the first coordinate.

Denote $e_i - e_j \in \mathbb{Z}^{2g+1}$ by $e_{i,j}$. The group $\mathbb{Z}^{2g}$ is the free abelian group on $\eta(e_{2,1}), \ldots, \eta(e_{2g+1,1})$, and so it has a presentation whose generators are $\eta(e_{2,1}), \ldots, \eta(e_{2g+1,1})$ and whose relations are $[\eta(e_{i,1}), \eta(e_{j,1})] = 1$.

We thus obtain a presentation for $\mathbb{Z}^{2g+1}_{bal}$ with generators $e_{2,1}, \ldots, e_{2g+1,1}$ and relations $[e_{i,1}, e_{j,1}] = 1$. If we add (formal) generators $e_{i,i}$ to this presentation, as well as relations $e_{i,i} = 1$, we obtain a new presentation for the same group; this is an elementary Tietze transformation [12, Section 1.5]. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Lemma 5.1, we have a short exact sequence

$$1 \to \ker \epsilon \to F^{even}_{2g+1} \xrightarrow{\epsilon} \mathbb{Z}^{2g+1}_{bal} \to 1$$

where $\epsilon(\zeta_i\zeta_j) = e_{i,1}$ and $\epsilon(\zeta_i^2) = e_{i,i} = 0$.

Consider the presentation for $\mathbb{Z}^{2g+1}_{bal}$ given in Lemma 5.2. The generators for that presentation are the images of the following generators for F^{even}_{2g+1}:

$$\{\zeta_1^2, \ldots, \zeta_{2g+1}^2\} \cup \{\zeta_2\zeta_1, \ldots, \zeta_{2g+1}\zeta_1\}.$$

If we lift each relator in the presentation for $\mathbb{Z}^{2g+1}_{bal}$ to the corresponding element of F^{even}_{2g+1}, we obtain a normal generating set for $\ker \epsilon$, that is, these elements and their conjugates in F^{even}_{2g+1} generate $\ker \epsilon$. The relators $e_{i,i}$ and $[e_{i,1}, e_{j,1}]$ lift to elements

$$\zeta_i^2 \quad \text{and} \quad [\zeta_i\zeta_1, \zeta_j\zeta_1],$$

respectively.

Passing through the isomorphism $F_{2g+1} \to \mathcal{SBK}(S_{g,2})$ from Theorem 3.2, and applying Theorem 1.2 we obtain a normal generating set for $\mathcal{SIBK}(S_{g,2})$.

Since the conjugate in $\mathrm{SMod}(S_{g,2})$ of a Dehn twist about a symmetric separating curve is another such Dehn twist, it remains to show that the image of each ζ_i^2 and $[\zeta_i\zeta_1, \zeta_j\zeta_1]$ in the group $\mathcal{SBK}(S_{g,2})$ can be written as a product of Dehn twists about symmetric separating curves.

To further simplify matters, the image of each ζ_i^2 in $\mathcal{SBK}(S_{g,2})$ is conjugate to ζ_1^2 in $\mathrm{SMod}(S_{g,2})$, and (up to taking inverses) the image of each $[\zeta_i\zeta_1, \zeta_j\zeta_1]$ is conjugate to $[\zeta_3\zeta_1, \zeta_2\zeta_1]$ in $\mathrm{SMod}(S_{g,2})$ (the point is that there are elements of $\mathrm{Mod}(S_{0,2g+2}, \overline{p})$ taking the elements ζ_1^2 and $[\zeta_3\zeta_1, \zeta_2\zeta_1]$ of $\pi_1(S_{0,2g+2}, \overline{p})$ to the other given elements). Thus, we are reduced to checking that the images in $\mathcal{SBK}(S_{g,2})$ of ζ_1^2 and $[\zeta_3\zeta_1, \zeta_2\zeta_1]$ are both products of Dehn twists about symmetric separating curves.

In the proof of Lemma 4.5, we showed that the image of ζ_1 in the group $\mathcal{SBK}(S_{g,2})$ is a half-twist about the arc β_1. It follows that the image of ζ_1^2 is the Dehn twist about the boundary of a regular neighborhood of β_1. This boundary is (isotopic to) a symmetric separating curve in $S_{g,2}$ cutting off a disk with two marked points.

It remains to analyze the element $[\zeta_3\zeta_1, \zeta_2\zeta_1]$. There is a closed disk in $S_{0,2g+2}$ that contains the distinguished marked point $\overline{p}$, the 1st, 2nd, and 3rd marked points of $S_{0,2g+2}$ (and no other marked points), and a representative of $[\zeta_3\zeta_1, \zeta_2\zeta_1] \in \pi_1(S_{0,2g+2}, \overline{p})$. Under the isomorphism $F_{2g+1} \to \mathcal{SBK}(S_{g,2})$ from Theorem 3.2, we see that the commutator $[\zeta_3\zeta_1, \zeta_2\zeta_1]$ maps to an element of $\mathcal{SI}(S_{g,2})$ supported on a copy of $S_{1,2}^1$ fixed by s. Any element of $\mathcal{SI}(S_{g,2})$ supported on $S_{1,2}^1$ restricts to an element of $\mathcal{SI}(S_{1,2}^1)$, and so Proposition 5.3 below implies that the image of $[\zeta_3\zeta_1, \zeta_2\zeta_1]$ in $\mathcal{SI}(S_{1,2}^1)$ is a product of Dehn twists about symmetric separating curves. Each symmetric separating curve in $S_{1,2}^1$ either cuts off a disk with two marked points or a genus 1 surface with two marked points, and so the image of $[\zeta_3\zeta_1, \zeta_2\zeta_1]$ in $\mathcal{SI}(S_{1,2}^1)$ is equal to a product of Dehn twists about such curves. It follows that the image of $[\zeta_3\zeta_1, \zeta_2\zeta_1]$ maps to an element of $\mathcal{SI}(S_{g,2})$ is equal to a product of Dehn twists about such curves, and we are done. □

In the proof of Theorem 1.4, we used the following fact, which is equivalent to a theorem of Smythe [19].

Proposition 5.3. *The group $\mathcal{SI}(S_{1,2}^1)$ is generated by Dehn twists about symmetric separating curves.*

Proof. By Theorem 4.2, we have

$$\mathcal{SI}(S_{1,2}^1) \cong \mathcal{SI}(S_{1,2}) \times \mathbb{Z},$$

where the $\mathbb{Z}$ factor is the Dehn twist about $\partial S_{1,2}^1$. Therefore, it suffices to show that $\mathcal{SI}(S_{1,2})$ is generated by Dehn twists about symmetric separating curves in $S_{1,2}$. This follows immediately from the fact that

$\mathrm{SMod}(S_{1,2}) = \mathrm{Mod}(S_{1,2})$ [8, Section 3.4] and the fact that the Torelli group of a torus with two marked points is generated by Dehn twists about separating curves (this can be proven directly via the argument of [2, Lemma 7.2], or it can be obtained immediately by combining [2, Lemma 7.2] with Lemma 5.8 below). $\square$

6. Application to closed surfaces

We now address Hain's conjecture that $\mathcal{SI}(S_g)$ is generated by Dehn twists about symmetric separating curves. Specifically, we will use Theorem 1.4 to prove Theorem 1.3, which states that, in order to prove Hain's conjecture, it is enough to show that $\mathcal{SI}(S_g)$ is generated by reducible elements.

6.1. **Reduction to the symmetrically reducible case.**

We say that an isotopy class a of simple closed curves is *symmetric* if it has a symmetric representative and it is *pre-symmetric* if it is not symmetric and $\sigma(a)$ and a have disjoint representatives. Also, we say that an element f of $\mathrm{SMod}(S_g)$ is *symmetrically strongly reducible* if there is an isotopy class of essential simple closed curves in S_g that is either symmetric or pre-symmetric and is preserved by f.

We have the following standard fact (see, e.g., [8, Lemma 2.9]). In the statement, we say that two simple closed curves α and β are in minimal position if $|\alpha \cap \beta|$ is minimal with respect to the homotopy classes of α and β.

Lemma 6.1. *Let S be any compact surface. Let α and β be two simple closed curves in S that are in minimal position and that are not isotopic. If $\phi : S \to S$ is a homeomorphism that preserves the set of isotopy classes $\{[\alpha], [\beta]\}$, then ϕ is isotopic to a homeomorphism that preserves the set $\alpha \cup \beta$.*

Proposition 6.2. *Let $g \geq 0$. If $f \in \mathcal{SI}(S_g)$ is reducible, then f is symmetrically strongly reducible.*

Proof. First of all, Ivanov proved that every reducible element of $\mathcal{I}(S_g)$ has the stronger property that it fixes the isotopy class of a single essential simple closed curve in S_g [10, Theorem 3]. As such we can choose an isotopy class a of simple closed curves in S_g that is fixed by f. We may assume that $\sigma(a) \neq a$, for in that case there is nothing to do. Since f lies in $\mathrm{SMod}(S_g)$, we have:

$$f(\sigma(a)) = \sigma(f(a)) = \sigma(a).$$

In other words, f fixes the isotopy class $\sigma(a)$. Since σ has order 2, it preserves the set of the isotopy classes $\{a, \sigma(a)\}$.

Let α and α' be representatives for a and $\sigma(a)$ that are in minimal position. Let μ denote the boundary of a closed regular neighborhood of $\alpha \cup \alpha'$, and let μ' denote the multicurve obtained from μ by deleting the inessential components of μ and replacing any set of parallel curves with a single curve. Lemma 6.1 implies that both f and σ fix the isotopy class of μ'. By Ivanov's theorem again, f fixes the isotopy class of each component of μ'.

Let $\mu_1, \ldots, \mu_k$ denote the connected components of μ'. If $k = 0$, that is to say a and $\sigma(a)$ fill S_g, then it follows that f has finite order (see [8, Proposition 2.8]); since $\mathcal{I}(S_g)$ is torsion free, f is the identity. Now suppose $k > 0$. By construction, $i(\mu_i, \mu_j) = 0$ for all i and j, and σ acts as an involution on the set of isotopy classes $\{[\mu_i]\}$. Thus, there is either a singleton $\{[\mu_i]\}$ or a pair $\{[\mu_i], [\mu_j]\}$ fixed by σ, and the proposition is proven. □

6.2. **Analyzing individual stabilizers.** Our next goal is to prove Proposition 6.6, which identifies the stabilizer in $\mathcal{SI}(S_g)$ of a symmetric or pre-symmetric curve with a product of hyperelliptic Torelli groups of surfaces with one or two marked points.

Let a be the isotopy class of an essential simple closed curve in S_g. Assume that a is symmetric or pre-symmetric. If a is symmetric and separating, we choose a representative simple closed curve α so that $s(\alpha) = \alpha$, and if a is nonseparating, we choose a representative simple closed curve α so that $s(\alpha) \cap \alpha = \emptyset$.

Let A denote either α or $\alpha \cup s(\alpha)$, depending on whether a is separating or nonseparating, respectively. Let R_1 and R_2 denote the closures in S_g of the two connected components of $S_g - A$. Let R_1' and R_2' denote the surfaces obtained from R_1 and R_2 obtained by collapsing each boundary component to a marked point. Let A' denote the set of marked points in either R_1' or R_2'.

Each pair (R_k', A') is homeomorphic to either $S_{g,1}$ or $S_{g,2}$. The hyperelliptic involution of S_g induces a hyperelliptic involution of each (R_k', A'). We can thus define $\mathrm{SMod}(R_k', A')$ and $\mathcal{SI}(R_k', A')$ as in Section 2.

We remark that if a is symmetric and nonseparating, then one of the surfaces (R'_k, A') is a sphere with two marked points. For this surface, $\mathrm{SMod}(R'_k, A') \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{SI}(R'_k, A') = 1$. For such a, it would have been more natural to take a representative α of a that is symmetric. However, the choice we made will allow us to make most of our arguments uniform for the various cases of a.

Let $\mathrm{SMod}(S_g, a)$ denote the stabilizer of the isotopy class a in $\mathrm{SMod}(S_g)$, and let $\mathrm{SMod}(S_g, \vec{a})$ denote the subgroup of $\mathrm{SMod}(S_g, a)$ consisting of elements that fix the orientation of a. We now define maps

$$\Psi_k : \mathrm{SMod}(S_g, \vec{a}) \to \mathrm{SMod}(R'_k, A')$$

for $k = 1, 2$.

Let $f \in \mathrm{SMod}(S_g, \vec{a})$, and let ϕ be a representative that commutes with s. We may assume that ϕ fixes α. Since ϕ commutes with s, it must also fix $s(\alpha)$. Since $f \in \mathrm{SMod}(S_g, \vec{a})$, it follows that ϕ does not permute the components of $S_g - A$, and hence induces a homeomorphism ϕ'_k of R'_k for each k. By construction, ϕ'_k commutes with the hyperelliptic involution of R'_k. Finally, we define

$$\Psi_k(f) = [\phi'_k].$$

We have the following standard fact; see [8, Proposition 3.20].

Lemma 6.3. *Let $g \geq 2$, and let a be either a symmetric or pre-symmetric isotopy class of simple closed curves in S_g. Define R'_i and A' as above. The homomorphism*

$$\Psi_1 \times \Psi_2 : \mathrm{SMod}(S_g, \vec{a}) \to \mathrm{SMod}(R'_1, A') \times \mathrm{SMod}(R'_2, A')$$

is well defined and has kernel

$$\ker(\Psi_1 \times \Psi_2) = \begin{cases} \langle T_a \rangle & a \ \ \textit{symmetric} \\ \langle T_a T_{\sigma(a)} \rangle & a \ \ \textit{pre-symmetric} \end{cases}$$

Let $\mathcal{SI}(S_g, a)$ denote $\mathcal{SI}(S_g) \cap \mathrm{SMod}(S_g, a)$. Since $\mathcal{SI}(S_g, a)$ is a subgroup of $\mathrm{SMod}(S_g, \vec{a})$, we can restrict each Ψ_k to $\mathcal{SI}(S_g, a)$.

Lemma 6.4. *Let $g \geq 2$. For $k \in \{1, 2\}$, the image of $\mathcal{SI}(S_g, a)$ under Ψ_k lies in $\mathcal{SI}(R'_k, A')$.*

Proof. By the relative version of the Mayer–Vietoris sequence, we have an exact sequence:

$$H_1(A, A) \to H_1(R_1, A) \times H_1(R_2, A) \to H_1(S_g, A) \to H_0(A, A).$$

(In this sequence, and in the rest of the proof, we take the coefficients for all homology groups to be $\mathbb{Z}$.) The first and last groups are trivial, and so we have

$$H_1(R_1, A) \times H_1(R_2, A) \cong H_1(S_g, A).$$

For each k, the map $R_k \to R'_k$ that collapses the boundary components to marked points induces an isomorphism

$$H_1(R_k, A) \cong H_1(R'_k, A').$$

The natural map $H_1(S_g) \to H_1(S_g, A)$ is not surjective in general (it fails to be surjective in the case that a is nonseparating). However, the composition

$$\pi : H_1(S_g) \to H_1(S_g, A) \stackrel{\cong}{\to} H_1(R'_1, A') \times H_1(R'_2, A') \to H_1(R'_k, A')$$

is surjective for $k \in \{1, 2\}$. Indeed, any element x of $H_1(R'_1, A') \cong H_1(R_1, A)$ is represented by a collection of closed oriented curves in R_1 and oriented arcs in R_1 connecting A to itself. If we connect the endpoints of each oriented arc in R_1 by a similarly oriented arc in R_2, we obtain an element of $H_1(S_g)$ that maps to x.

By construction the following diagram is commutative:

$$\begin{tikzcd}
H_1(S_g) \arrow[r, "f_\star"] \arrow[d, "\pi"'] & H_1(S_g) \arrow[d, "\pi"] \\
H_1(R'_k, A') \arrow[r, "\Psi_k(f)_\star"] & H_1(R'_k, A').
\end{tikzcd}$$

The lemma follows immediately. □

Let $\hat{\imath}(\cdot, \cdot)$ denote the algebraic intersection form on $H_1(S_g; \mathbb{Z})$.

Lemma 6.5. *Let $g \geq 2$, and let a and b be isotopy classes of oriented simple closed curves in S_g. Suppose that a is pre-symmetric, b is symmetric, and $\hat{\imath}([a], [b])$ is odd. Let $k \in \mathbb{Z}$. If $[b] + k[a]$ is represented by a symmetric simple closed curve, then k is even.*

Proof. Let $V \cong (\mathbb{Z}/2\mathbb{Z})^{2g+1}$ denote the mod 2 homology of the surface obtained from $S_{0,2g+2}$ by removing the marked points. Choosing a distinguished marked point in $S_{0,2g+2}$ gives a natural basis for V. Arnol'd proved that the branched cover $S_g \to S_{0,2g+2}$ induces an isomorphism of $H_1(S_g; \mathbb{Z}/2\mathbb{Z})$ with the kernel of the map $V \to \mathbb{Z}/2\mathbb{Z}$ obtained by summing coordinates [1, Lemma 2]. Under this identification, the algebraic intersection number on $H_1(S_g; \mathbb{Z}/2\mathbb{Z})$ descends to the usual dot product on V. Also, the elements of $H_1(S_g)$ with symmetric representatives are exactly the ones mapping to elements of V with exactly two or $2g$ nonzero entries. Elements of $H_1(S_g)$ with pre-symmetric representatives map to elements of V that fail this property. The result follows easily. □

Proposition 6.6. *Let $g \geq 2$, and let a be the isotopy class of a simple closed curve in S_g that is either symmetric or pre-symmetric. Define A' and R'_k as above. The homomorphism*

$$(\Psi_1 \times \Psi_2)|_{\mathcal{SI}(S_g,a)} : \mathcal{SI}(S_g, a) \to \mathcal{SI}(R'_1, A') \times \mathcal{SI}(R'_2, A')$$

is surjective with kernel

$$\ker(\Psi_1 \times \Psi_2)|_{\mathcal{SI}(S_g,a)} = \begin{cases} \langle T_a \rangle & a \text{ is separating} \\ 1 & a \text{ is nonseparating.} \end{cases}$$

Proof. The kernel of $(\Psi_1 \times \Psi_2)|_{\mathcal{SI}(S_g,a)}$ is $\ker(\Psi_1 \times \Psi_2) \cap \mathcal{SI}(S_g, a)$. The description of $\ker(\Psi_1 \times \Psi_2)|_{\mathcal{SI}(S_g,a)}$ in the statement of the lemma then follows from Lemma 6.3.

By Lemma 6.4, we have

$$(\Psi_1 \times \Psi_2)(\mathcal{SI}(S_g, a)) \subseteq \mathcal{SI}(R'_1, A') \times \mathcal{SI}(R'_2, A').$$

It remains to show that $(\Psi_1 \times \Psi_2)|_{\mathcal{SI}(S_g,a)}$ is surjective. Let $f' \in \mathcal{SI}(R'_1, A') \times \mathcal{SI}(R'_2, A')$. Choose some $f \in \mathrm{SMod}(S_g, \vec{a})$ that maps to f'.

Fix some orientation of a. Consider the natural map

$$\eta : H_1(S_g; \mathbb{Z})/\langle [a] \rangle \to H_1(R'_1, A'; \mathbb{Z}) \times H_1(R'_2, A'; \mathbb{Z}).$$

The mapping classes f and f' induce automorphisms $f_\star$ and $f'_\star$ of $H_1(S_g; \mathbb{Z})$ and $\mathrm{Im}(\eta)$, respectively. Since $f_\star([a]) = [a]$, we further have that $f_\star$ induces an automorphism $\bar{f}_\star$ of $H_1(S_g; \mathbb{Z})/\langle [a] \rangle$.

If we give a an orientation, then it represents an element of $H_1(S_g; \mathbb{Z})$. There is a commutative diagram:

$$\begin{array}{ccccc} H_1(S_g;\mathbb{Z})/\langle [a] \rangle & \xrightarrow{\bar{f}_\star} & H_1(S_g;\mathbb{Z})/\langle [a] \rangle & & \\ \downarrow \eta & & \downarrow \eta & & \\ \mathrm{Im}(\eta) & \xrightarrow{f'_\star} & \mathrm{Im}(\eta) & \subset & H_1(R'_1, A'; \mathbb{Z}) \times H_1(R'_2, A'; \mathbb{Z}). \end{array}$$

Since $f'_\star$ is the identity and η is injective, it follows that $\bar{f}_\star$ is the identity.

If a is separating, then $[a] = 0$ and so $f_\star = \bar{f}_\star$ is the identity. Thus, f is an element of $\mathcal{SI}(S_g, a)$, and since $\Psi_1 \times \Psi_2$ maps f to f', we are done in this case.

If a is nonseparating, we can find an isotopy class b of oriented symmetric simple closed curves in S_g with $\hat{\imath}([a],[b]) = 1$. Since $\bar{f}_\star$ is the identity, we have

$$f_\star([b]) = [b] + k[a]$$

for some $k \in \mathbb{Z}$.

Our next goal is to find some $h \in \ker(\Psi_1 \times \Psi_2)$ so that $(hf)_\star$ fixes $[b]$. If a is symmetric, then we can simply take h to be T_a^k (cf. [8, Proposition 8.3]). If a is pre-symmetric, then this does not work, since $T_a^k \notin \mathrm{SMod}(S_g)$. However, if a is pre-symmetric, then Lemma 6.5 implies that k is even. Thus we can take h to be $\left(T_a T_{\sigma(a)}\right)^{k/2}$.

Now, let x be any element of $H_1(S_g;\mathbb{Z})$. Since $\bar{f}_\star$ is the identity and since h induces the identity map on $H_1(S_g;\mathbb{Z})/\langle [a] \rangle$ we have $(hf)_\star(x) = x + j[a]$ for some $j \in \mathbb{Z}$. Since $(hf)_\star$ is an automorphism of $H_1(S_g;\mathbb{Z})$, we have:

$$\begin{aligned} \hat{\imath}(x,[b]) &= \hat{\imath}((hf)_\star(x), (hf)_\star([b])) \\ &= \hat{\imath}(x + j[a], [b]) \\ &= \hat{\imath}(x,[b]) + j\,\hat{\imath}([a],[b]) \\ &= \hat{\imath}(x,[b]) + j \end{aligned}$$

and so $j = 0$. Thus $(hf)_\star(x) = x$ and so $hf \in \mathcal{SI}(S_g, a)$. Since $h \in \ker(\Psi_1 \times \Psi_2)$, we have that $(\Psi_1 \times \Psi_2)(hf) = f'$, and we are done. □

6.3. **Finishing the proof.** In Section 6.1 we showed that reducible elements of $\mathcal{SI}(S_g)$ are strongly symmetrically reducible, and in Section 6.2 we studied strongly symmetrically reducible elements of $\mathcal{SI}(S_g)$. We now combine the results from these sections with our Birman exact sequences for $\mathcal{SI}(S_g)$ in Section 3 in order to prove Theorem 1.3.

Proof of Theorem 1.3. As in the statement of the theorem, we assume that $\mathcal{SI}(S_k)$ is generated by Dehn twists about symmetric separating curves for all $k < g$. Let $f \in \mathcal{SI}(S_g)$ be a reducible element. When g is 1, there is nothing to do, since $\mathcal{SI}(S_g)$ is trivial, so we may assume $g \geq 2$.

By Proposition 6.2, f is symmetrically strongly reducible. In other words, there is an isotopy class a of essential simple closed curves in S_g, where a is either symmetric or pre-symmetric, and where $f \in \mathcal{SI}(S_g, a)$.

Define A', R_1', and R_2' as in Section 6.2. As per Proposition 6.6, there is a (surjective) homomorphism

$$(\Psi_1 \times \Psi_2)|_{\mathcal{SI}(S_g)} : \mathcal{SI}(S_g, a) \to \mathcal{SI}(R_1', A') \times \mathcal{SI}(R_2', A'),$$

and each element of the kernel is a power of a Dehn twist about a symmetric separating curve (when a is nonseparating, the kernel is trivial). For $i = 1, 2$, each Dehn twist about a symmetric separating simple closed curve in $\mathcal{SI}(R_i', A')$ has a preimage in $\mathcal{SI}(S_g, a)$ that is also a Dehn twist about a symmetric separating curve. Thus, to prove the theorem, it suffices to show that each element of $\mathcal{SI}(R_i', A')$ is a product of Dehn twists about symmetric separating curves.

Fix $i \in \{1, 2\}$. Say that R_i' has genus g_i. Note that $0 \leq g_i < g$. Combining Theorem 1.1 with Theorem 1.4, we have a short exact sequence

$$1 \to \mathcal{SIBK}(R_i', A') \to \mathcal{SI}(R_i', A') \to \mathcal{SI}(S_{g_i}) \to 1,$$

where each element of $\mathcal{SIBK}(R_i', A')$ is a product of Dehn twists about symmetric separating simple closed curves in (R_i', A') (in the case that a is separating, we have $\mathcal{SIBK}(R_i', A') = 1$). Recall we are assuming that $\mathcal{SI}(S_{g_i})$ is generated by Dehn twists about symmetric separating curves. Since each such Dehn twist has a preimage in $\mathcal{SI}(R_i', A')$ that is also a Dehn twist about a symmetric separating curve, it follows that each element of $\mathcal{SI}(R_i', A')$ is a product of Dehn twists about symmetric separating curves, and we are done. $\square$

6.4. **Small twists.** Finally, we prove Proposition 1.5, which states that every Dehn twist in $\mathcal{SI}(S_g)$ is a product of Dehn twists about symmetric separating curves of genus 1 and 2. In particular, this tells us that Hain's conjecture, that $\mathcal{SI}(S_g)$ is generated by Dehn twists about symmetric separating curves, implies that $\mathcal{SI}(S_g)$ is in fact generated by Dehn twists about symmetric separating curves of genus 1 and 2.

Proof of Proposition 1.5. We prove by induction that any Dehn twist about a symmetric separating curve of genus k is equal to a product of Dehn twists about symmetric separating curves of genus 1 and 2. For $k \leq 2$, there is nothing to do, so assume that $k \geq 3$. Let c be a symmetric separating curve of genus k. Then let d be a symmetric separating curve of genus $k - 1$ lying on the side of c homeomorphic to

S_k^1, and let a be a symmetric nonseparating curve in S_g lying between c and d. It suffices to show that $T_c T_d^{-1}$ is equal to a product of Dehn twists about symmetric separating curves of genus 1 and 2.

At this point, we proceed in a similar fashion to the proof of Theorem 1.3. Considering $T_c T_d^{-1}$ as an element of $\mathcal{SI}(S_g, a)$, define the subsets $A', R_1', R_2' \subset S_g$ as in Section 6.2. Since a is symmetric and nonseparating, we can assume that R_1' and R_2' have genus $g-1$ and 0, respectively. In particular, $\mathcal{SI}(R_2', A')$ is trivial. Denote the two points of A' by p_1 and p_2. Since $R_1' \cong S_{g-1}$, Proposition 6.6 then gives that there is an isomorphism $\mathcal{SI}(S_g, a) \to \mathcal{SI}(S_{g-1,2})$.

The image in $\mathcal{SI}(S_{g-1,2})$ of the product $T_c T_d^{-1}$ lies in the kernel of the forgetful map

$$\mathcal{SI}(S_{g-1,2}) \to \mathcal{SI}(S_{g-1}).$$

By Theorem 1.4, the image of the product $T_c T_d^{-1}$ in $\mathcal{SI}(S_{g-1,2})$ is equal to a product of Dehn twists about symmetric separating curves, each of which cuts off either a disk with two marked points or a torus with two marked points. The preimages of these two kinds of twists in $\mathcal{SI}(S_g, a)$ are Dehn twists about symmetric separating curves of genus 1 and 2, respectively. Thus, $T_c T_d^{-1}$ equals a product of such Dehn twists. $\square$

It is also true for $\mathcal{SI}(S_g^1)$ and $\mathcal{SI}(S_g^2)$ that each Dehn twist is a product of Dehn twists about symmetric separating curves of genus 1 and 2 (thus, there is an analogous theorem for $\mathcal{BI}_n$). To prove this, we need to strengthen Theorem 1.4 so that it also covers the cases of $S_{g,2}^1$ and $S_{g,2}^2$ (the latter has one pair of interior marked points and one pair of boundary components, both interchanged by s). The main difference in these cases is that the quotient surface is a disk instead of a sphere. This is not crucial because the fundamental group is still F_{2g+1}, and if we treat the leftmost marked point in Figure 2 as a boundary component, we can use the same generating set as before. After making this adjustment, the proof is essentially the same.

References

[1] V. I. Arnol′d. A remark on the branching of hyperelliptic integrals as functions of the parameters. *Funkcional. Anal. i Priložen.*, 2(3):1–3, 1968.

[2] Mladen Bestvina, Kai-Uwe Bux, and Dan Margalit. The dimension of the Torelli group. *J. Amer. Math. Soc.*, 23(1):61–105, 2010.

[3] Joan S. Birman. Mapping class groups and their relationship to braid groups. *Comm. Pure Appl. Math.*, 22:213–238, 1969.

[4] Joan S. Birman. On Siegel's modular group. *Math. Ann.*, 191:59–68, 1971.

[5] Joan S. Birman and Hugh M. Hilden. On the mapping class groups of closed surfaces as covering spaces. In *Advances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, pages 81–115. Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.

[6] Tara Brendle, Leah Childers, and Dan Margalit. Cohomology of the hyperelliptic Torelli group. *Israel J. Math., to appear.*

[7] Tara Brendle, Dan Margalit, and Andrew Putman. Generators for the hyperelliptic Torelli group and the kernel of the Burau representation at $t = -1$. arXiv:1211.4018.

[8] Benson Farb and Dan Margalit. *A primer on mapping class groups*. Princeton University Press, 2011.

[9] Richard Hain. Finiteness and Torelli spaces. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 57–70. Amer. Math. Soc., Providence, RI, 2006.

[10] Nikolai V. Ivanov. *Subgroups of Teichmüller modular groups*, volume 115 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by E. J. F. Primrose and revised by the author.

[11] Dennis Johnson. The structure of the Torelli group. II. A characterization of the group generated by twists on bounding curves. *Topology*, 24(2):113–126, 1985.

[12] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory*. Dover Publications Inc., New York, revised edition, 1976. Presentations of groups in terms of generators and relations.

[13] Wilhelm Magnus and Ada Peluso. On a theorem of V. I. Arnol′d. *Comm. Pure Appl. Math.*, 22:683–692, 1969.

[14] Geoffrey Mess. The Torelli groups for genus 2 and 3 surfaces. *Topology*, 31(4):775–790, 1992.

[15] Takayuki Morifuji. On Meyer's function of hyperelliptic mapping class groups. *J. Math. Soc. Japan*, 55(1):117–129, 2003.

[16] Bernard Perron. Mapping class group and the Casson invariant. *Ann. Inst. Fourier (Grenoble)*, 54(4):1107–1138, 2004.

[17] Jerome Powell. Two theorems on the mapping class group of a surface. *Proc. Amer. Math. Soc.*, 68(3):347–350, 1978.

[18] Andrew Putman. Cutting and pasting in the Torelli group. *Geom. Topol.*, 11:829–865, 2007.

[19] N. F. Smythe. The Burau representation of the braid group is pairwise free. *Arch. Math. (Basel)*, 32(4):309–317, 1979.

[20] Barbara van den Berg. *On the abelianization of the Torelli group*. PhD thesis, Universiteit Utrecht, 2003.

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