

WEIL–PETERSSON ISOMETRIES VIA THE PANTS COMPLEX

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ABSTRACT. We extend a theorem of Masur and Wolf which states that given a finite area hyperbolic surface S , every isometry of the Teichmüller space for S with the Weil–Petersson metric is induced by an element of the mapping class group for S . Our argument handles the previously untreated cases of the four times-punctured sphere, the once-punctured torus, and the twice-punctured torus.

1. INTRODUCTION

The purpose of this paper is to fill in several gaps in the understanding of Weil–Petersson isometries. Throughout, S is an orientable finite area hyperbolic surface, and $S_{g,n}$ denotes a surface of genus g with n punctures. The *extended mapping class group* for S is:

$$\text{Mod}(S) = \pi_0(\text{Homeo}^\pm(S))$$

Let $T(S)$ denote the Teichmüller space of S with the Weil–Petersson metric. Masur and Wolf proved that for $S \notin \{S_{1,1}, S_{0,4}, S_{1,2}\}$, every isometry of $T(S)$ is induced by an element of $\text{Mod}(S)$ [18]. One of the main ingredients is that for exactly these surfaces, every automorphism of the complex of curves for S is induced by $\text{Mod}(S)$; this is a theorem of Ivanov [12], with special cases due to Korkmaz and Luo [14] [15]. See Section 2 for definitions.

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We are able to extend the result of Masur and Wolf to all possible surfaces by viewing an isometry I of $T(S)$ as giving an automorphism \bar{I}_* of the *pants complex* for S . By a theorem of the second author, \bar{I}_* is necessarily induced by a mapping class f . It then follows from a theorem of Wolpert, which builds upon a theorem of the first author, that f induces I . Thus, we have:

Main Theorem. *The natural map $\eta : \text{Mod}(S) \rightarrow \text{Isom}(T(S))$ is surjective. Further, $\ker(\eta) \cong \mathbb{Z}_2$ for $S \in \{S_{1,1}, S_{1,2}, S_{2,0}\}$, $\ker(\eta) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $S = S_{0,4}$, $\ker(\eta) = \text{Mod}(S)$ for $S = S_{0,3}$ and $\ker(\eta) = 1$ otherwise.*

It follows from the definition of the Weil–Petersson metric that $\text{Mod}(S)$ acts by isometries; the content is that all isometries arise in this way.

The application of curve complexes to the study of isometries of Teichmüller space first appears in Ivanov’s proof of Royden’s Theorem, which is the analogue of our Main Theorem for the Teichmüller metric [13] [20].

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2. PRELIMINARIES

We briefly recall the basic definitions and concepts.

Complex of curves. The *complex of curves* $C(S)$ for S , defined by Harvey [8], is the abstract simplicial flag complex with vertices corresponding to isotopy classes of simple closed curves in S , and edges between vertices that can be realized disjointly in S .

For our purposes, this complex has an undesirable property when S is $S_{1,1}$, $S_{0,4}$, or $S_{1,2}$: there are automorphisms of $C(S)$ which are not

induced by elements of $\text{Mod}(S)$. For $S \in \{S_{1,1}, S_{0,4}\}$, this is because, as defined, $C(S)$ is a countable discrete set of vertices with no edges.

The problem with $S_{1,2}$ is more subtle. Luo noticed that if ι is the hyperelliptic involution of $S_{1,2}$, then the projection $\pi : S_{1,2} \rightarrow S_{1,2}/\iota$ gives an isomorphism between $C(S_{1,2})$ and $C(S_{0,5})$. To see the isomorphism, one must identify $S_{0,5}$ with the complement of a closed regular neighborhood of the cone points in $S_{1,2}/\iota$. One must also apply the fact, due to Birman and Viro, that ι acts trivially on $C(S_{1,2})$ [2] [22]. Since $\text{Mod}(S_{0,5})$ acts transitively on the vertices of $C(S_{0,5})$ (every curve has two punctures on one side and three on the other), it follows that some automorphisms of $C(S_{1,2})$ fail to preserve the set of vertices corresponding to separating curves. Such automorphisms clearly cannot arise from mapping classes (see [15]).

Pants complex. The *pants complex* $P(S)$ for S , introduced by Hatcher and Thurston [10] [9], is the simplicial complex with vertices corresponding to *pants decompositions* of S (i.e. maximal simplices of $C(S)$), and edges connecting pants decompositions which differ by an *elementary move*, by which we mean that the two pants decompositions differ by only one curve, and the differing curves have the smallest possible intersection. The minimal intersection number is 1 or 2, depending on whether the curve being replaced lies in a single pair of pants or is the boundary between two pairs of pants; see Figure 1.

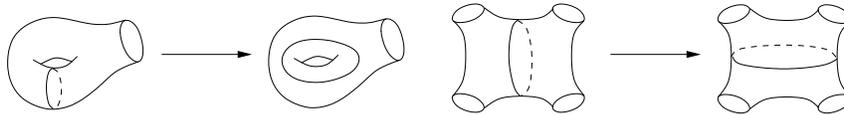


FIGURE 1. Elementary moves.

We will apply the following theorem of the second author [16]. We use $\text{Aut}(P(S))$ to denote the simplicial automorphism group of $P(S)$.

Theorem 2.1. *The natural map $\theta : \text{Mod}(S) \rightarrow \text{Aut}(P(S))$ is surjective. Further, $\ker(\theta) \cong \mathbb{Z}_2$ for $S \in \{S_{1,1}, S_{1,2}, S_{2,0}\}$, $\ker(\theta) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $S = S_{0,4}$, $\ker(\theta) = \text{Mod}(S)$ for $S = S_{0,3}$, and $\ker(\theta) = 1$ otherwise.*

The proof of Theorem 2.1 is based on the aforementioned theorem of Ivanov, Korkmaz, and Luo, which gives an analogous statement for $C(S)$. We remark that the term “pants complex” is often reserved for a particular simply-connected 2-complex obtained by adding 2-cells to

the graph defined above [9]. However, we ignore this distinction, as Theorem 2.1 holds for either complex.

Teichmüller space. A point in the *Teichmüller space* $T(S)$ of S is given by a pair (X, f) , where X is a finite-area hyperbolic surface X and $f : S \rightarrow X$ is a homeomorphism. Two points (X, f) and (Y, g) are equivalent if $g \circ f^{-1}$ is isotopic to an isometry.

We note that $T(S_{0,3})$ is a single point, and in this case our main result is trivial.

Weil–Petersson metric. A point in $T(S)$ is naturally a Riemann surface via its uniformization as a quotient of \mathbb{H}^2 by a Fuchsian group. In this way, the cotangent space $T_X^*(T(S))$ at a point X is identified with the space of holomorphic quadratic differentials on X (holomorphic forms of type $\phi(z) dz^2$, with simple poles at the punctures of X), which has the L^2 inner product defined by:

$$\langle \phi, \psi \rangle = \int_X \frac{\phi \bar{\psi}}{\rho^2}$$

Here, $\rho(z)|dz|$ is the hyperbolic metric on X . Then, the *Weil–Petersson metric* is defined by the pairing

$$\langle \mu, \phi \rangle = \int_X \mu \phi$$

where $\phi \in T_X^*(T(S))$ and μ is an element of the tangent space $T_X(T(S))$, which is identified with the space of harmonic Beltrami differentials on X (forms of type $\mu(z) \overline{dz}/dz$; see e.g. [11] [19]).

Chu and Wolpert showed that the Weil–Petersson metric is not complete [6] [23].

Augmented Teichmüller space. Masur gave an interpretation of points in the completion of $T(S)$ as *marked noded Riemann surfaces* [17]. A *noded Riemann surface* is a complex space with at most isolated singularities, called *nodes*, each possessing a neighborhood biholomorphic to a neighborhood of $(0, 0)$ in the curve $\{zw = 0\}$ in \mathbb{C}^2 . Removing the nodes of a noded Riemann surface W yields a (possibly disconnected) Riemann surface whose components (which are required to have negative Euler characteristic) we call the *pieces* of W . The notion of a noded Riemann surface was developed by Bers [1].

Given a simplex $\sigma \in C(S)$, a *marked noded Riemann surface* (W, f) with nodes at σ is a noded Riemann surface W equipped with a continuous mapping

$$f: S \rightarrow W$$

so that $f|_{S \setminus \sigma}$ is a homeomorphism to the pieces of W . Two marked noded Riemann surfaces (W, f) and (Z, g) are equivalent if there is a continuous marking-preserving map $\phi: W \rightarrow Z$ which preserves nodes and is biholomorphic on the pieces.

To describe a neighborhood of a point (W, f) in Masur’s completion of $T(S)$, which we denote $\overline{T(S)}$, we give coordinates adapted to the simplex σ . Given a maximal simplex τ with $\sigma \subset \tau$, there are *Fenchel–Nielsen coordinates*

$$\{(\ell, \theta)_\alpha\} \in (\mathbb{R}_{>0} \times \mathbb{R})^{|\tau|}$$

for $T(S)$ specifying length and twist parameters along each closed geodesic α determined by the vertices of τ (see, e.g. [11]). Then the *extended Fenchel–Nielsen coordinates* for τ are obtained by allowing lengths to range in $\mathbb{R}_{\geq 0}$ and stipulating that for any twist parameters θ and θ' we have $(0, \theta)_\alpha \sim (0, \theta')_\alpha$.

The extended Fenchel–Nielsen coordinates for (W, f) , then, are given by setting the length parameters corresponding to the curves of σ to 0, and setting the other length and twist parameters equal to the corresponding Fenchel–Nielsen coordinates for the piece in which the geodesic lies. Then a neighborhood of (W, f) is given by all surfaces in $\overline{T(S)}$ with extended Fenchel–Nielsen coordinates close to those of (W, f) . We will in practice refer to (W, f) as a noded surface with curves in σ *pinched*.

Stratification. We think of $\overline{T(S)}$ as a stratified space, with \mathcal{F}_k denoting the stratum consisting of surfaces with k curves pinched. Also, \mathcal{F} denotes $\overline{T(S)} - T(S)$. Points in the stratum corresponding to maximally noded surfaces are naturally associated with the pants decompositions that are pinched. These points are isolated from each other, since there is a unique hyperbolic structure on $S_{0,3}$.

As Teichmüller spaces themselves, each connected component \mathcal{O} of \mathcal{F}_k comes equipped with its own Weil–Petersson metric $d_{\mathcal{O}}$. On the other hand, \mathcal{O} inherits a metric $d_{T(S)}|_{\mathcal{O}}$ as the completion of $T(S)$. Masur proved that the Weil–Petersson metric tensor on $T(S)$ extends continuously to the intrinsic Weil–Petersson metric tensor on strata in

\mathcal{F} [17], and Daskalopoulos–Wentworth and Wolpert proved that length minimizing paths in \mathcal{F}_k do not enter $T(S)$ [7] [24]. Together these facts give the following (see [18, Lem. 1.3.1] [7] [24]):

Theorem 2.2. *On any connected component \mathcal{O} of \mathcal{F}_k , the metrics $d_{\mathcal{O}}$ and $d_{T(S)}|_{\mathcal{O}}$ are the same.*

Negative curvature. By work of Royden, Tromba, and Wolpert the Weil–Petersson metric has negative sectional curvatures and is geodesically convex [21] [24]. It then follows from general principles that $\overline{T(S)}$ is a CAT(0) metric space (see [3]).

Visual sphere. Each point X in $T(S)$ has a *visual sphere* $\mathcal{V}_X(S)$, which is the unit tangent space at X . Due to the non-completeness of the Weil–Petersson metric, certain directions in the visual sphere correspond to finite-length geodesics emanating from X that leave every compact subset of $T(S)$. These *finite rays* terminate at noded Riemann surfaces in \mathcal{F} .

As a consequence of the CAT(0) property for $\overline{T(S)}$, the first author proved the following density theorem [4].

Theorem 2.3. *For any S and any $X \in T(S)$, the finite rays are dense in the visual sphere $\mathcal{V}_X(S)$.*

The idea is that every point X of $T(S)$ is within a uniformly bounded distance of a maximally noded surface $N(X)$ (see [5]). Thus, given a sequence of points X_n diverging from X along an infinite ray from X , a sequence points of $\mathcal{V}_X(S)$ corresponding to the directions determined by the geodesics joining X to $N(X_n)$ converges to the given infinite ray. Wolpert used Theorem 2.3 to prove the following theorem [25], which immediately implies that a Weil–Petersson isometry is determined by its action on the maximally noded surfaces.

Theorem 2.4. *$\overline{T(S)}$ is the closed convex hull of its maximally noded surfaces.*

Wolpert has in fact already used this corollary to give a simplification of the proof of Masur and Wolf (but not our extension) [25].

3. EDGE STRATA

In this section, we study the stratum of $\overline{\mathbb{T}(S)}$ where all but one of the curves of a pants decomposition are pinched; in Lemma 4.2, this stratum is where we will be able to see the edges of $P(S)$. By Theorem 2.2, each connected component of this stratum is isometric to $\mathbb{T}(R)$ for R equal to either $S_{1,1}$ or $S_{0,4}$.

In either case, we have that $\mathbb{T}(R)$ is topologically \mathbb{H}^2 , and $\text{Mod}(R)/Z \cong \text{PGL}_2(\mathbb{Z})$, where Z is the center of $\text{Mod}(R)$ ($Z = \mathbb{Z}_2$ if $R = S_{1,1}$ and $Z = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $R = S_{0,4}$). The action of $\text{Mod}(R)$ on $\mathbb{T}(R)$ then coincides with the usual action of $\text{PGL}_2(\mathbb{Z})$ on \mathbb{H}^2 by Möbius transformations and complex conjugation.

Also, $\mathcal{F} = \overline{\mathbb{T}(R)} - \mathbb{T}(R)$ is the set of rational points of $\partial\mathbb{H}^2$, and $\overline{\mathbb{T}(R)}$ has the horoball topology. That \mathcal{F} is discrete agrees with the fact that any noded surface is maximally noded (pants decompositions consist of exactly one curve). In this way, curves in R can be identified with the rational numbers, and two curves p/q and r/s (in reduced form) are connected by an edge in $P(R)$ exactly when $|ps - qr| = 1$.

3.1. Farey graph. Let \mathcal{G} denote the usual embedding of the *Farey graph* in \mathbb{H}^2 : in the upper half-plane model of \mathbb{H}^2 , its vertices are at rational points (including ∞), and its edges are hyperbolic geodesics connecting (reduced) rational numbers p/q and r/s whenever $|ps - qr| = 1$ (see Figure 2). We will now see that each edge of \mathcal{G} is a Weil–Petersson geodesic.

The imaginary axis is a geodesic since it is the fixed set of the isometry given by a mapping class corresponding to:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which acts on \mathbb{H}^2 (in the upper half-plane model) by $z \mapsto -\bar{z}$. The $\text{PGL}_2(\mathbb{Z})$ -orbit of this geodesic is \mathcal{G} . Thus, \mathcal{G} gives a geodesic between any noded surfaces whose corresponding pants decompositions have distance 1 in $P(R)$.

3.2. Weil–Petersson vs. pants distance. We will now show that the Weil–Petersson metric encodes adjacency in the pants complex $P(R)$.

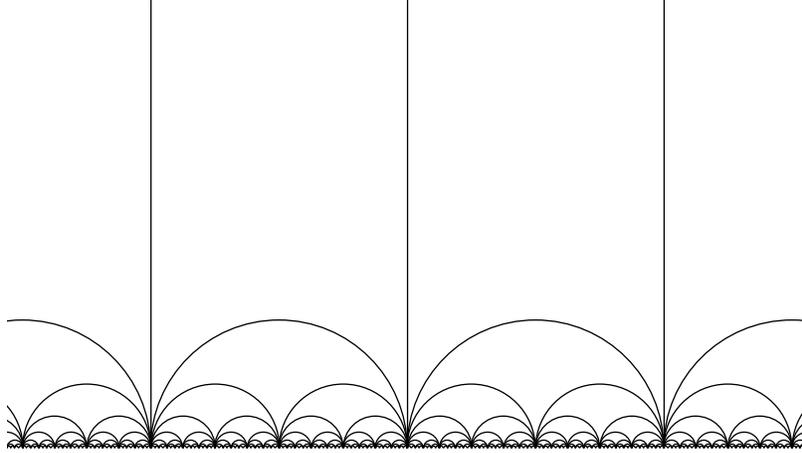


FIGURE 2. The Farey graph \mathcal{G} in the upper half-plane.

In the following lemma, we denote a (geodesic) triangle, and any of its edges, by a set of vertices. A triangle is called a *tripod* if any edge is contained in the union of the other two edges.

Lemma 3.1. *Let (X, d) be a CAT(0) metric space, and let ABC be an equilateral triangle in X which is not a tripod, and which has side length L . If D is any point on BC , then $d(A, D) > L/2$.*

Proof. Assume that, say, $d(B, D) \leq L/2$. The triangle inequality implies that $d(A, D) \geq L/2$ and that if $d(A, D) = L/2$, then $d(B, D) = d(C, D) = L/2$. In the latter case, the uniqueness of geodesics in CAT(0) spaces implies that $ADC = AC$ and $ADB = AB$; thus, ABC is a tripod.

□

For the next lemma, L is the Weil–Petersson length of edges of \mathcal{G} , d_P denotes combinatorial distance in $P(R)$, and d_{WP} denotes Weil–Petersson distance.

Lemma 3.2. *If P and P' are pants decompositions of R with $d_P(P, P') > 1$, then the corresponding noded surfaces W and W' in $T(R)$ have $d_{WP}(W, W') > L$.*

Proof. Let g be the geodesic from W to W' . Since $d_P(P, P') > 1$, g passes through at least two triangles of \mathcal{G} , and this gives a natural

division of g into segments. By Lemma 3.1, the segments containing W and W' each have length greater than $L/2$, so $d_{\text{WP}}(W, W') > L$.

□

Remark. L has different values, say $L_{S_{1,1}}$ and $L_{S_{0,4}}$, depending on the surface; in fact, $L_{S_{0,4}} = 2L_{S_{1,1}}$ and more generally the metrics for $\mathbb{T}(S_{0,4})$ and $\mathbb{T}(S_{1,1})$ differ by the same factor. To see this, note that for R either $S_{1,1}$ or $S_{0,4}$, the quotient of R by $Z(\text{Mod}(R))$ is a genus zero orbifold with one cusp and three cone points of order 2. Since $\pi : R \rightarrow R/Z(\text{Mod}(R))$ is a local isometry, and π gives a canonical bijection between $\mathbb{T}(R)$ and $\mathbb{T}(R/Z(\text{Mod}(R)))$, it follows from the definition that the metric on $\mathbb{T}(R)$ is the metric on $\mathbb{T}(R/Z(R))$ multiplied by the degree of π . This fact, however, is not needed for the proof.

4. PROOF OF MAIN THEOREM

Let S be any surface, other than $S_{0,3}$, with negative Euler characteristic, let $I \in \text{Isom}(\mathbb{T}(S))$, and let \bar{I} be the natural extension of I to the completion $\overline{\mathbb{T}(S)}$. We will use $\mathcal{F}_{\geq k}$ to denote $\bigcup_{i \geq k} \mathcal{F}_i$.

The following lemma is also proven by Masur and Wolf [18, Lem. 1.3.4, Lem. 1.3.5]; however, the proof given here is more elementary, relying only on topology and not at all on curvature. The lemma also follows from earlier work of Wolpert, who proved that the connected components of strata are characterized as being connected unions of open geodesics [24, Thm. 13].

Lemma 4.1. $\bar{I}(\mathcal{F}) \subset \mathcal{F}$, I and \bar{I} are surjective, and \bar{I} preserves strata.

Proof. For the first statement, suppose that $\bar{I}(W) \in \mathbb{T}(S)$. Then $\bar{I}(W)$ has a compact neighborhood K . Let $K' = K \cap \bar{I}(\overline{\mathbb{T}(S)})$. We have that K' is compact, since $\bar{I}(\overline{\mathbb{T}(S)})$ is closed (it is an isometric embedding of a complete space). But this implies that W has a compact neighborhood (namely $\bar{I}^{-1}(K')$), and so $W \in \mathbb{T}(S)$.

For surjectivity of I , it suffices to show that I is proper (any proper embedding of a manifold without boundary into itself is surjective). Indeed, let K be a compact subset of $\mathbb{T}(S)$. As above, $K' = K \cap \bar{I}(\overline{\mathbb{T}(S)})$ is compact. Since $\bar{I}(\mathcal{F}) \subset \mathcal{F}$, $K' = K \cap I(\mathbb{T}(S))$. Hence, $I^{-1}(K')$ is

compact. Thus, I is surjective, and it immediately follows that \bar{I} is surjective.

For the last statement, we inductively show that $W \in \mathcal{F}_k$ if and only if $\bar{I}(W) \in \mathcal{F}_k$. The base case is $\mathcal{F}_0 = \mathrm{T}(S)$: by definition, $\bar{I}(W) \in \mathrm{T}(S)$ whenever $W \in \mathrm{T}(S)$, and we have just shown that $\bar{I}(\mathcal{F}) \subset \mathcal{F}$.

Now suppose \bar{I} preserves $\mathcal{F}_1, \dots, \mathcal{F}_{k-1}$. Then \bar{I} restricts to an isometry of $\mathcal{F}_{\geq k}$, which is a homeomorphism since \bar{I} is surjective. The points of \mathcal{F}_k are characterized by having compact neighborhoods in $\mathcal{F}_{\geq k}$. Since $\bar{I}|_{\mathcal{F}_{\geq k}}$ is a homeomorphism, \mathcal{F}_k is preserved.

□

We now combine Lemmas 3.2 and 4.1 to bridge the gap between isometries of $\mathrm{T}(S)$ and automorphisms of $\mathrm{P}(S)$. In the proof, we use \mathcal{F}_M to denote the highest level stratum (the 0-dimensional stratum) of $\mathrm{T}(S)$. Also, each connected component \mathcal{O} of \mathcal{F}_{M-1} (the 2-dimensional stratum) comes equipped with a Farey graph, which we denote by $\mathcal{G}_{\mathcal{O}}$.

Lemma 4.2. \bar{I} induces $\bar{I}_\star \in \mathrm{Aut}(\mathrm{P}(S))$.

Proof. Points of \mathcal{F}_M correspond to vertices of $\mathrm{P}(S)$. Thus, by Lemma 4.1, we have that \bar{I} induces an automorphism \bar{I}_\star of the 0-skeleton of $\mathrm{P}(S)$.

It remains to show that \bar{I}_\star preserves edges of $\mathrm{P}(S)$. If \mathcal{O} is any connected component of \mathcal{F}_{M-1} , then \mathcal{O} and $\bar{I}(\mathcal{O})$ are both copies of either $\mathrm{T}(S_{1,1})$ or $\mathrm{T}(S_{0,4})$, and the edges of $\mathcal{G}_{\mathcal{O}}$ and $\mathcal{G}_{\bar{I}(\mathcal{O})}$ correspond to edges of $\mathrm{P}(S)$.

The Weil–Petersson lengths of the edges of $\mathcal{G}_{\mathcal{O}}$ and $\mathcal{G}_{\bar{I}(\mathcal{O})}$ must be the same, since, by Lemma 4.1, the smallest length of a geodesic connecting noded surfaces is an isometry invariant. By Lemma 3.2, Lemma 4.1 and Theorem 2.2, $\bar{I}(\mathcal{G}_{\mathcal{O}}) = \mathcal{G}_{\bar{I}(\mathcal{O})}$; but this is exactly saying that $\bar{I}_\star \in \mathrm{Aut}(\mathrm{P}(S))$.

□

We note that, in the proof, the surfaces corresponding to \mathcal{O} and $\bar{I}(\mathcal{O})$ must be the same; this follows from the remark at the end of Section 3.

We are now ready to prove the Main Theorem.

Proof. Let $I \in \text{Isom}(\mathbb{T}(S))$. By Lemma 4.2, I induces $\bar{I}_* \in \text{Aut}(\mathbb{P}(S))$. By Theorem 2.1, \bar{I}_* is induced by a mapping class f . By Theorem 2.4, f induces I .

Also, by Theorem 2.4, the kernel of the map $\eta : \text{Mod}(S) \rightarrow \text{Isom}(\mathbb{T}(S))$ is the same as the kernel of $\theta : \text{Mod}(S) \rightarrow \text{Aut}(\mathbb{P}(S))$.

□

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