On Congruence Subgroups of the Braid Group
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**Goal:** Understand the structure of congruence subgroups of the braid group.

![Braids](image)

<table>
<thead>
<tr>
<th>Integral Burau Representation</th>
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<tbody>
<tr>
<td>ρ⁻¹ : Bₙ → GL(n, ℤ)</td>
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<tr>
<td>σᵢ ↦ Iᵢ⁻¹ ⊕ [ \begin{array}{cc} 2 &amp; -1 \ -1 &amp; 0 \end{array} ] ⊕ Iₙ⁻ᵢ⁻¹</td>
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<tr>
<td>rₙ is the usual mod N reduction map</td>
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<td>Bₙ[N] = ker(rₙ ∘ ρ⁻¹)</td>
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**Problem I: Generating Sets**

**Question:** What is a natural generating set for Bₙ[4]? How big is it?

Margalit and Kordek: Size lower bounded by

\[
\left( \binom{n}{2} + 3\binom{n}{3} + 3\binom{n}{4} \right) \sim O(n^4)
\]

Schreier’s method \(\sim\) exponential generating set
Use recurrence relation to reduce generating set

**Theorem.**

# generators of \(Bₙ[4]\) \(\sim\) \(O(n^5)\)

**Problem II: \(PBₙ^{\ell}\) and \(Bₙ[2\ell]\)**

**Question:** What is the relationship between \(PBₙ^{\ell}\) and \(Bₙ[2\ell]\) for varying \(\ell\)?

Brendle and Margalit: \(PBₙ^{\ell} = Bₙ[4]\)

**Theorem.**

For \(\ell = 2^k\), \(PBₙ^{\ell} \subseteq Bₙ[2\ell]\)

For \(\ell = 6, 10, 12\) or \(\ell\) odd, \(PBₙ^{\ell} \not\subseteq Bₙ[2\ell]\)

**Conjecture.**

\(\ell = 2^k \iff PBₙ^{\ell} \subseteq Bₙ[2\ell]\)

**Problem III: Quotients**

**Question:** What can we say about quotients of Burau levels?

Artin: \(Bₙ/PBₙ \cong S_n\)
Stylianakis: \(Bₙ[p]/Bₙ[2p] \cong S_n\) for \(p\) prime

**Theorem.**

\(Bₙ[\ell]/Bₙ[2\ell] \cong S_n\) for odd \(\ell\)
\(Bₙ[\ell]/Bₙ[2\ell] \cong (\mathbb{Z}/2\mathbb{Z})^{(\ell)}\) for even \(\ell\)

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