Breaking the square-root barrier in the parity of the prime counting function

August 28, 2009

1 Overview

Let us assume that we can compute

$$D(x) \ := \ \sum_{n \leq x} \tau(n)$$

in time at most $x^{1/2-\epsilon}$. Then, of course, this means we can compute the parity of the sum

$$P(x) \ := \ \pi(x) + \sum_{d \geq 2} \pi(x/d^2)$$

in time at most

$$x^{1/2-\epsilon} + x^{1/4+o(1)},$$

which is at most about $x^{1/2-\epsilon}$ for $\epsilon < 1/4$, which we assume that it is.

Using this, we will show that we can compute the parity of just $\pi(x)$ in time at most

$$x^{1/4+1/(4+16\epsilon/3)}.$$

2 The algorithm

We observe that

$$\sum_{e^2 \leq x^{1-\delta}} P(x/e^2) \ = \ \pi(x) + A + B,$$

where

$$A := \sum_{4\le n^2\le x^{1-\delta}} \pi(x/n^2) \sum_{\substack{e^2\le x^{1-\delta}\\ e^2|n^2}} 1 \;=\; \sum_{4\le n^2\le x^{1-\delta}} \tau(n)\pi(x/n^2)$$

$$\equiv \sum_{1<m^4\le x^{1-\delta}} \pi(x/m^4) \pmod 2.$$

and where

$$B := \sum_{x^{1-\delta}<n^2\le x} \pi(x/n^2) \sum_{\substack{e^2\le x^{1-\delta}\\ e^2|n^2}} 1 \;=\; \sum_{\substack{p\le x^\delta\\ p\ \text{prime}}} \sum_{x^{1-\delta}<n^2\le x/p} \sum_{\substack{e^2\le x^{1-\delta}\\ e^2|n^2}} 1$$

$$= \sum_{\substack{p\le x^\delta\\ p\ \text{prime}}} \sum_{\substack{x^{1/2-\delta/2}<ef\le\sqrt{x/p}\\ e\le x^{1/2-\delta}}} 1.$$

Now, for each $p \le x^\delta$, the inner sum on B can be computed in time at most $(x/p)^{1/4}$ using the hyperbola method. So, B can be computed using at most about

$$x^{1/4}\sum_{p\le x^\delta} 1/p^{1/4} \;\ll\; x^{1/4+3\delta/4}$$

bit operations.

We also have, by our assumption, that the time it takes to compute the above sum of $P(x/e^2)$ is at most about

$$\sum_{e^2\le x^{1-\delta}} (x/e^2)^{1/2-\epsilon} \;<\; x^{1/2-\epsilon} \sum_{e\le x^{1/2-\delta/2}} 1/e^{1-2\epsilon} \;\ll\; x^{1/2-\epsilon\delta}.$$

Thinking ahead about how to minimize our run time later on, we will choose $\delta > 0$ so that the time it takes to compute B is comparable to computing the sum of $P(x/e^2)$; so, we will choose $\delta > 0$ to satisfy

$$1/2 - \epsilon\delta \;=\; 1/4 + 3\delta/4 \quad\Longrightarrow\quad \delta = 1/(3+4\epsilon).$$

And so, we will aim for an overall running time to compute $\pi(x)$ of something like

$$x^{1/4+1/(4+16\epsilon/3)}.$$

Now, of course, if we were able to compute the parity of $\pi(y)$, $y < x/16$, in time about $y^{1/4+1/(4+16\epsilon/3)}$, then we should be able to compute the parity A above in time about $x^{1/4+1/(4+16\epsilon/3)}$; so, putting everything together, we are done.