Braid index, the fractional Dehn twist coefficients and Upsilon

Diana Hubbard (joint w/ Peter Feller)

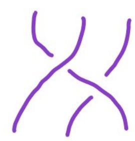
A **knot** is a smooth embedding $S^1 \hookrightarrow S^3$

link " " $S^1 \cup \dots \cup S^1 \hookrightarrow S^3$

Braids are representatives of knots/links

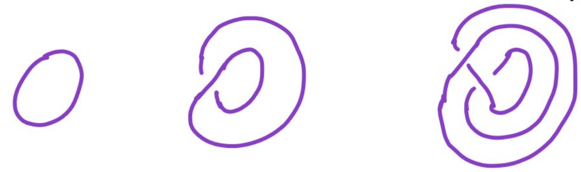
Braid group on n strands

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \,\middle|\, \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\rangle$$

$\sigma_1 \sigma_2^{-1} \in B_3$  $\xrightarrow{\text{closure}} \widehat{\sigma_1 \sigma_2^{-1}}$ 

closure of any braid is a knot/link

Alexander (1923): Every knot or link can be realized as the closure of some braid (in fact infinitely many)

ex 

Def$^{\underline{n}}$: the braid index of a knot or link K is the minimal n so that K is the closure of n-braid

A way to mesure the "complexity" of a knot.

Question: What is known about the braid index?

Answer: Not a lot

A famous result:

Morton-Franks-Williams inequality '87

let d_+, d_- the max/min degree in v in HOMFLY poly of K

the braid index b_K of K satisfies

$$\tfrac{1}{2}(d_+ - d_-) + 1 \leq b_K$$

Jones '87: On all but 5 of the knots upto 10 crossings this is sharp

Kawamuro '06: Fails to be sharp for ∞ many & the defect can be arbitrarily large

Corollary of MFW:

if a knot K has an n-braid rep β where $\beta = \Delta^2 \alpha$ then K has braid index n

α ← positive

Δ^2 → special braid "full twist" $(\sigma_1 \ldots \sigma_{n-1})^n$

Braids as mapping classes

B_n is the $\underbrace{\text{mapping class group}}$ of n-punctured disk D_n

↪ group of orientation pres. homeo of D_n that fix ∂D_n / isotopies that fix ∂D_n

every element ϕ in MCG of a surface with one boundary component can be assigned its fractional Dehn twist coeff (FDTC) that measures the amount of twisting ϕ effects about the boundary

ex: in B_3 $\quad \tau(\Delta^2) = 1 \quad \tau(\sigma_1\sigma_2) = 1/3 \quad \tau(\sigma_1) = 0$

$\Delta^2 = (\sigma_1\sigma_2)^3 \qquad \tau(\sigma_1\sigma_2^{-1}) = 0$

Idea of def^n: compactify the universal cover $\widetilde{D}_n$ of D_n to get $\overline{D}_n$ use the action of the lift of β on $\partial\overline{D}_n$ to construct a map $\theta : B_n \to \mathrm{Homeo}^+(\mathbb{R})$

define $\tau(\beta) =$ translation number of $\theta(\beta)$

$|\tau(\alpha\beta) - \tau(\alpha) - \tau(\beta)| \leq 1$

$\tau(\alpha^n) = n\,\tau(\alpha)$

Thm 1 (Feller–H.)

fix $n \geq 2$, Any n-braid β with $|\tau(\beta)| > n-1$ realizes the braid index of its closure

This is close to optimal: the bound could at best be improved to $n-2$, $\exists$ examples of n-braids w/ FDTC $n-2$ that do not realize the braid index of their closures (Malyatin–Netsvetaev)

recall for MFW: $\beta = \Delta^2 \alpha$

α positive

ex: $\beta = (\Delta^2)^4 (\sigma_1 \sigma_2)^{-k} \in B_4$

k arbitrarily large $\quad \tau(\beta) = 4$

Tools that go into the proof

- A left-order on B_n (due to Dehornoy)
- A characterization of FDTC in terms of order (Malyutin)
- A calculation of Υ, a knot concordance invariant due to Ozsváth–Stipsicz–Szabó for torus knots
- A characterization of FDTC in terms of $\tilde{\Upsilon}$, a braid quasimorphism arising from Υ (Brandenbursky)
- Generalized Jones conjecture due to Dynnikov–Prasolov, Lafountain–Menasco

(2013) consider n-braid β, m-braid α

$\hat{\beta} = \hat{\alpha}$ and m is the braid index of $\hat{\alpha}$

Then $|wr(\beta) - wr(\alpha)| \leq n - m$

wr: exponent sum

A left order on B_n (Dehornoy)

$\beta > 1$ if it can be written as a σ_i-positive word for some i

↳ no σ_j^{-1}'s up to i

σ_i only appears positively

$\sigma_2 \sigma_3^{-1000} > 1$

$\sigma_3^{-1000} \sigma_2 > 1$

$\alpha < \beta$ if $\alpha^{-1}\beta > 1$

for any β, $\exists!$ m st. $(\Delta^2)^{m+1} > \beta \geq (\Delta^2)^m$

$m = \lfloor \beta \rfloor$

<u>Malyutin '04</u>

$$\tau(\beta) = \lim_{k\to\infty} \frac{\lfloor \beta^k \rfloor}{k}$$

<u>Th$^{\underline{m}}$ (Feller-H)</u>:

Fix $n \geq 2$. If an n-braid β satisfies $\beta \geq \Delta^{2n}$ or $\beta \leq \Delta^{-2n}$ then β realizes braid index of its closure

<u>Concordance</u>: Two links K, L are <u>concordant</u> if $\exists$ ori. smooth embedding of disjoint annuli in $S^3 \times [0,1]$ s.t. ori. boundary is $K \times \{0\} \cup L \times \{1\}$

Ozsváth, Stipsicz, Szabó: associate to concordance class of a knot K a PL function

$$\Upsilon_K : [0,1] \to \mathbb{R}$$

<u>Brandenbursky '11</u>: The following is well-defined

$$\tilde{\Upsilon} : B_n \to \mathrm{Cont}\,[0,1]$$

$$\beta \longmapsto \lim_{k\to\infty} \frac{\Upsilon_{\widehat{\beta^k \varepsilon_{\beta^k}}}}{k}$$

where ε_{β^k} is a shortest possible word s.t. $\widehat{\beta^k \varepsilon_{\beta^k}}$ a knot

Th^{m} 3:

Fix $n \geq 2$ for any n-braid β we have

$$\tilde{\Upsilon}_\beta(t) = \begin{cases} -t\,\dfrac{wr(\beta)}{2} & t \leq \frac{2}{n} \\ -t\,\dfrac{wr(\beta)}{2} + \tau(\beta)\,n\left(t - \frac{2}{n}\right) & \frac{2}{n} \leq t \leq \frac{2}{n-1} \end{cases}$$

(Feller–Krcatovich)

Putting it all together to prove thm 1

Jones conj: $\underset{n}{\beta}\,,\ \underset{m}{\alpha}$ braid index $\hat{\beta} = \hat{\alpha}$

$$|wr(\beta) - wr(\alpha)| \leq n - m$$

Prop (Feller-H) $\hat{\beta}$ concordant to $\hat{\alpha}$

$$|\tilde{\Upsilon}_\beta(t) - \tilde{\Upsilon}_\alpha(t)| \leq t\,\frac{n-1+m-1}{2}$$

$t \in [0,1]$

⋮