Sheaf theoretic models for $SL(2,\mathbb{C})$ Floer homology

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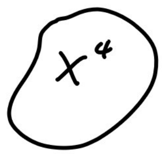
(joint w/ M. Abouzaid)

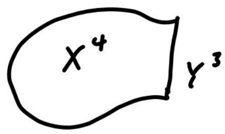
Review of $SU(2)$ Gauge Theory (1980's)

Donaldson



study $\underbrace{\text{ASD}}_{SU(2)}$ Yang-Mills eq^{n}s $\Big\downarrow$ count sol^{n}s

$D_X \in \mathbb{Z}$ detect smooth structures



$D(x) \in I_x(Y) = H_*(C_*, \partial)$ $\quad (H_1(Y)=0)$

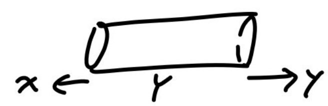
(Floer) instanton homology

C_* = gen'd by (perturbed) irreducible flat $SU(2)$ connections over Y

$\{\pi_1(Y) \to SU(2)\}^{irr}/conj$

$\partial x = \sum n_{xy} y$

n_{xy} = # sol^{n}s ASD eq^{n}s on $\mathbb{R}\times Y$



Taubes: $\chi(I_*(Y)) = 2\lambda(Y)$

Casson invt.

Goal: do this for $SL(2,\mathbb{C})$ instead of $SU(2)$

"non-compact gauge theory"

Motivation:

① $\pi_1(Y) \to SL(2,\mathbb{C})$ are more closely related to 3-manifold topology, hyperbolic geometry

② $SL(2,\mathbb{C})$ ASD YM eq^{n}s on X^4 $\overset{}{\underset{\text{deform}}{\rightsquigarrow}}$ Kapustin-Witten eq^{n}s

Conj (Witten 2011):

a) count sol^{n}s to KW equations on $\mathbb{R}^3\times[0,\infty)$ w/ boundary conditions depending on $K \subseteq \mathbb{R}^3$

b) count sol$^{\underline{n}}$s to Haydys-Witten eq$^{\underline{n}}$s on $\mathbb{R}^3 \times [0,\infty) \times \mathbb{R}$

$\rightsquigarrow$ Khovanov homology of K $Kh(K)$

<u>Open</u>: Extend Kh to knots $K \subset Y^3$ instead of $\mathbb{R}^3$

Want to study KW, HW equations on $Y \times [0,\infty)$

$\rightsquigarrow$ noncompactness issues (cf. Taubes)

<u>Easier</u>: study KW on $Y \times \mathbb{R} \rightsquigarrow SL(2,\mathbb{C})$ Floer homology

Why is this tractable?

model for $SU(2)$ Floer: Morse homology for $h: M \to \mathbb{R}$

C_k gen. by $\underset{\overset{\Downarrow}{x}}{crit(h)}$ of index k

$\partial x = \sum n_{xy} y$ ← count flow lines of ∇h

M compact $H_*(C_*, \partial) = H_*(M)$

model for $SL(2,\mathbb{C})$ Floer: Morse homology for $\mathrm{Re}\, h$ $\quad h: M^{2n} \to \mathbb{C}$ holomorphic

$x \in Crit(\mathrm{Re}\, h)$ locally $h = z_1^2 + \ldots + z_n^2$

$\mathrm{Re}(h) = \sum x_i^2 - y_i^2$

so index n

all crit pts in degree n

$\rightsquigarrow \partial = 0$

If we have N isolated irreducible $SL(2,\mathbb{C})$ flat conn on Y

$\rightsquigarrow$ except $SL(2,\mathbb{C})$ Floer: $\mathbb{Z}^N$

$$\chi^{irr}(Y) = \{\pi_1(Y) \to SL(2,\mathbb{C})\}^{irr}/_{conj}$$

can be arbitrarily complicated (cf. Kapovich-Millson)

if you perturb $\partial \neq 0$

<u>Goal</u>: define $SL(2,\mathbb{C})$ Floer homology for Y^3 in some way

<u>idea</u>: follow <u>Atiyah-Floer</u> <u>Conjeture</u>: gauge theory $\rightsquigarrow$ symplectic geom

for $SU(2)$: $Y^3 = U_0 \cup_{\Sigma^2} U_1$ Heegaard splitting

$\chi^{SU(2)}_{irr}(U_0)$ $\chi^{SU(2)}_{irr}(U_1) \subset \chi^{SU(2)}_{irr}(\Sigma)$

Lagrangian symplectic

intersect in $\chi^{SU(2)}_{irr}(Y)$

<u>A-F Conj</u>: $I_*(Y) = HF(L_0, L_1)$ (cf. Daemi-Fukaya)

Lagrangian Floer

<u>Do this for $SL(2,\mathbb{C})$</u>: $L_0 = \chi^{SL(2,\mathbb{C})}_{irr}(U_0)$, $L_1 = \chi^{SL(2,\mathbb{C})}_{irr}(U_1)$ $\subset M = \chi^{SL(2,\mathbb{C})}_{irr}(\Sigma)$

complex Lagrangian

complex symplectic mfd ω - holomorphic 2-form

$$L_0 \cap L_1 = \chi^{SL(2,\mathbb{C})}_{irr}(Y)$$

D. Joyce et al.
V. Bussi

$$HF(\text{complex Lagrangian}) \overset{conj}{=} \underbrace{\mathbb{H}^*}_{\text{hypercohomology}}(\underbrace{P_{L_0,L_1}})$$

perverse sheaf of vanishing cycles on $L_0 \cap L_1$

complex of constructable sheaves

locally: $L_0 = \mathbb{C}^n$
$L_1 = \text{graph}(dh)$
$h: L_0 \to \mathbb{C}$

stalk of sheaf:
$\tilde{H}^{*+\text{shift}}(\underbrace{h^{-1}(\varepsilon) \cap B(\delta)}_{\text{Milnor fiber}})$

Apply this to $SL(2,\mathbb{C})$ character varieties

<u>Th$^{\underline{m}}$ (Abouzaid-M)</u>

closed, oriented Y^3

$Y = U_0 \cup_\Sigma U_1$ let $HP^*(Y) = \mathbb{H}^*(P^\circ_{L_0,L_1})$

$\in Perv(\chi_{SL(2,\mathbb{C})}(Y))$

Then $P^\circ_{L_0,L_1}$ is a natural invariant of Y

hence so is $HP^*(Y)$

<u>Pf</u>: check: - invariance under stabilization of Heegaard splittings

- naturality (handleswap) ✓

$SL(2,\mathbb{C})$ "full" Casson invt : $\lambda^P(Y) = \chi(HP^*(Y))$

$\neq$ $SL(2,\mathbb{C})$ Casson invt. of Curtis $\lambda_{SL(2,\mathbb{C})}(Y)$

counted isolated irr. flat $SL(2,\mathbb{C})$ conns

Examples: 1) $\pi_1(Y)$ abelian $\Rightarrow HP^*(Y) = 0$ b/c $\chi_{irr}(Y) = \emptyset$

(S^3, Lens spaces, T^3, ...)

2) $Y = \Sigma(p,q,r)$ Brieskorn sphere : $\chi_{irr}(Y) = N$ isolated points

$$N = \frac{(p-1)(q-1)(r-1)}{4}$$

$HP^* = \mathbb{Z}^N$ (degree 0)

3) $\Sigma(2,3,5,7)$ has 23 isolated irred. to $SL(2,\mathbb{C})$

6 families, smooth $\dim_{\mathbb{C}} = 2$

cohomology $\mathbb{Z}\ 0\ \mathbb{Z}^5\ 0\ 0$

$$P^{\circ}_{L_0,L_1} = \begin{cases} \mathbb{Z} & \text{degree 0 over pts} \\ \mathbb{Z} & \text{degree } -2 \text{ over families} \end{cases}$$

$HP^* =$

deg	-2	-1	0
	$\mathbb{Z}^6$	0	$\mathbb{Z}^{23} \oplus \mathbb{Z}^{5\times 6}$

$\lambda_P = 23 + 36 = 59$

$\lambda_{Curtis} = 23$

Prop: Y^3 irred, not Haken $\Rightarrow HP^*(Y)$ is in degree 0

Pf: Culler-Shalen $\chi^{SL(2,\mathbb{C})}_{irr}(Y)$ is 0-dim'l

Fact: We can define HP^* for all cx semisimple Lie groups

e.g. $SL(n,\mathbb{C})$

Future goals:
- do this for $K \subset Y$, knot surgery formula?
- invts of 4-dim'l mfd w/ boundary : 3+1 TQFT (Kapustin-Witten)
- categorify this $\rightsquigarrow$ associate a category to Y^3

$\Downarrow$

related to Khovanov homology