Planar Graphs and Legendrian Surfaces

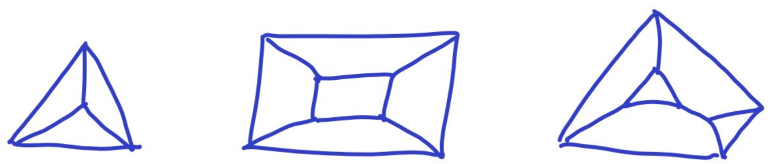
Emmy Murphy

based on work of Treumann-Zaslow
Casals-M-Sackel

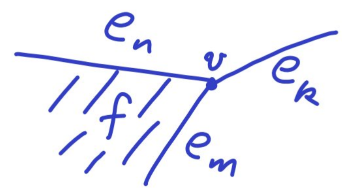
let $G \subseteq \mathbb{R}^2$ be a $\underbrace{\text{cubic}}_{\text{valence } 3}$ planar graph

$\Rightarrow \exists$ some $g \geq 0$ s.t.
$$|v| = 2g+3$$
$$|e| = 3g+3$$
$$|f| = g+3$$

<u>examples</u>:



for each face f, assign formal expression
$$\partial f = \sum_{v \in f} e_n e_m e_k^{-1}$$



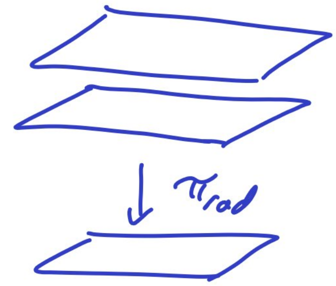
let $\mathbb{F} = \mathbb{F}_q$ be the field with $q = p^s$ elts
$$X = \{(e_1, \dots, e_{3g+3}) \in (\mathbb{F}^*)^{3g+3} : \partial f_j = 0 \ \forall j\}$$

<u>Th$^{\underline{m}}$ (Sackel)</u>:
$$\# X = \chi_{G^*}(q+1) \frac{(q-1)^{g+2}}{q^2+q}$$

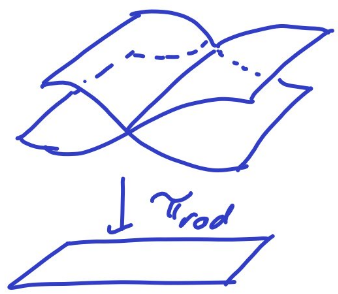
where $\chi_{G^*}(q+1) = \#$ of face colorings w/ $q+1$ colors

From G going to build a singular surface $\Sigma_G \subseteq \mathbb{R}^3$ (T-Z)
Σ_G is going to live in a small nbhd of $S^2 \subseteq \mathbb{R}^3$
and $\pi_{rad} : \Sigma_G \to S^2$ will be a 2:1 branched cover w/ branching locus $= v$

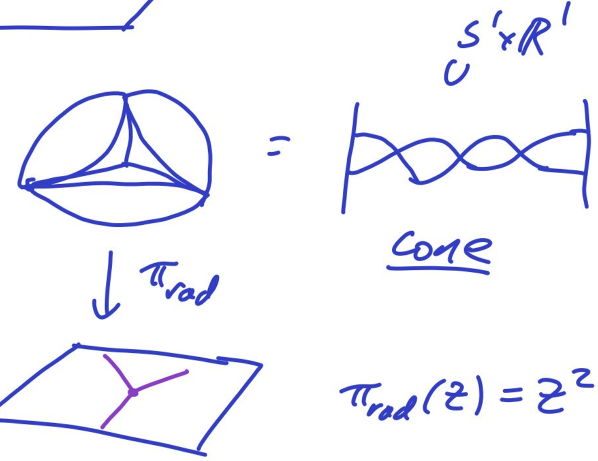
near faces of G: $\Sigma_G =$



near edges of G: $\Sigma_G =$



near verticies of G: $\Sigma_G =$



Have $\Sigma_G \subseteq S^3$

$$\left\{ \sum_{i=1}^{4} z_i^2 = 1 \right\} \subseteq \mathbb{C}^4$$

$\quad \cong$ symplectomorphic

T^*S^3 (real part S^3)

consider conormal bundle $\nu^* \Sigma_G \subseteq T^* S^3$

$= \{$all covectors which vanish on "$T\Sigma_G$"$\}$

$\mathbb{R}$ bundle over Σ_G

it is a 3-manifold in T^*S^3

$$i(T\nu^*\Sigma_G) \perp T\nu^*\Sigma_G$$

i.e. maximally far away from being holomorphic

i.e. locally equivalent to $\mathbb{R}^3 \subseteq \mathbb{C}^3$
(Lagrangian)

Manifolds of this form make good boundary conditions for holomorphic curves

i.e. S a Riemann sfc with boundary
look at maps $S \xrightarrow{u} T^*S^3$
s.t. u is holomorphic
$u|_{\partial S} : \partial \Sigma \to \nu^* \Sigma_G$

this is elliptic: solutions come in finite dimensional families

What is the topology of Σ_G?

Σ_G is a closed surface of genus g

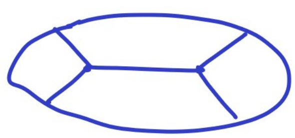
$\nu^* \Sigma_G \cap S^3 = \Sigma_G$ ← real S^3

$[(T^*S^3) \setminus S^3] \cap \nu^* \Sigma_G$ is smooth

here we use valance 3 of G

$\nu^* \Sigma_G - \Sigma_G \cong_{C^\infty} \Sigma_G \times \mathbb{R}$

given e_i = edge in G
e_i is a class in $H_1(\Sigma_g)$



double branched cover is annulus

<u>Idea</u>: Count all rigid holomorphic disks in $T^*S^3 \setminus S^3$ with ∂ on $\nu\Sigma_G \setminus \Sigma_G$ and also keep track of its homology class in $\nu^*\Sigma_G \setminus \Sigma_G$

so 1 holo curve for each face/vertex adjacency

so ∂f from before: all curves w/ asymptotics to f w/ homology coeff why is this recovering graph colorings?

look at space of rank 1 constructible sheaves on $\mathbb{R}^3$ w/ singular support on $\nu^* \Sigma_G$

$\cong$ chromatic info of G^*

Larger conjecture (Nadler-Zaslow)

in general constructible sheaves

$\Longleftrightarrow$

homs from holomorphic Floer theories