Lightning Talks I
Tech Topology Conference

December 6, 2019

Braid Index of Knotted Surfaces

Sudipta Kolay

Georgia Tech

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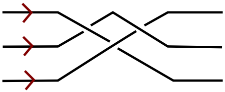
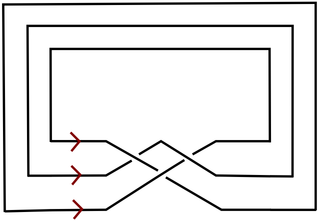


Braid

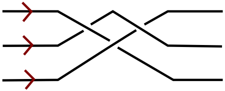


Closed Braid

Question

Is every link a closed braid?

Closing up the ends of a braid gives a link, called a *closed braid*.



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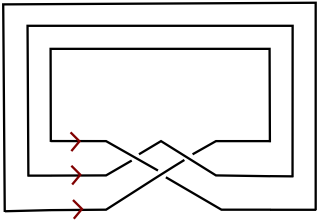

Closed Braid

Alexander's Theorem (1923)

Every oriented link in S^3 is isotopic to a closed braid.

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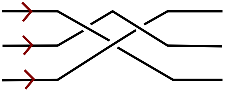

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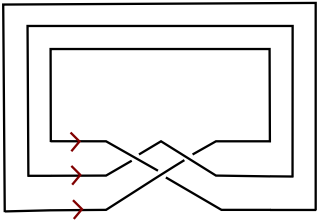

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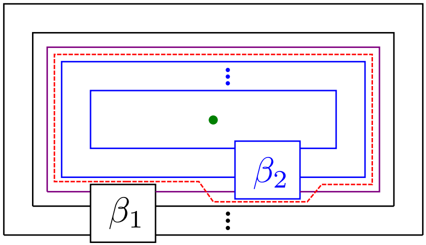
The *braid index* of a link L in $\mathbb{R}^3$ is the minimum number of strands required to express it as a closed braid, denoted $\text{Braid}(L)$.

Braid index under connect sum and generalized cabling

Theorem (Birman-Menasco 1990)

For knots K_1, K_2 in S^3,

$$\text{Braid}(K_1 \# K_2) = \text{Braid}(K_1) + \text{Braid}(K_2) - 1.$$



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Theorem (Williams 1992)

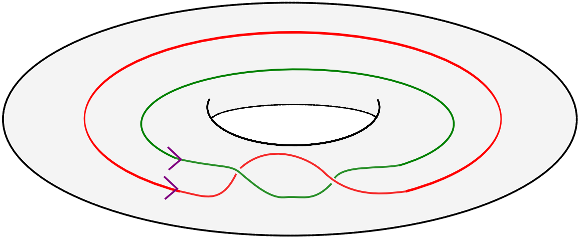
For a non-trivial knot K in $\mathbb{R}^3$, with pattern a closed n-braid

$$\text{Braid}(P(K)) = n \cdot \text{Braid}(K).$$

Closed braid in higher dimensions

Definition

An embedding $f : M^k \hookrightarrow S^k \times D^2 \subset S^{k+2}$ will be called a *closed braid* if $pr_1 \circ f : M^k \to S^k$ is a branched covering map.



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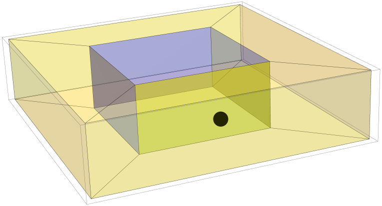
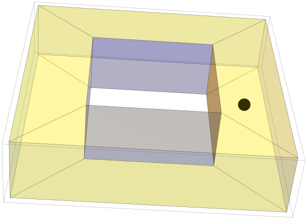
Analogues of Alexander's Theorem

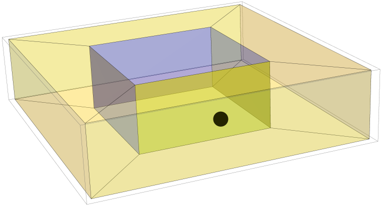
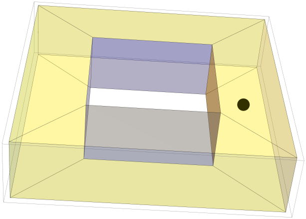
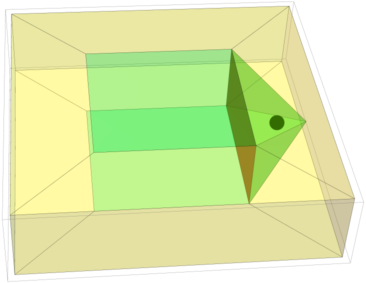
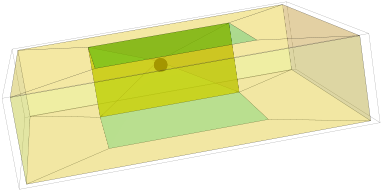
For $n \leq 5$, every codimension two embedded orientable submanifold in S^n is isotopic to a closed braid.

- smooth ribbon surfaces in $\mathbb{R}^4$, Rudolph (1983).
- $k = 2$ Viro (1990), Kamada (1994).
- $k = 3$ in the *PL category*, K. (*2017*).





Braiding the standard torus

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For non-trivial surface knots K_1, K_2 in S^4,

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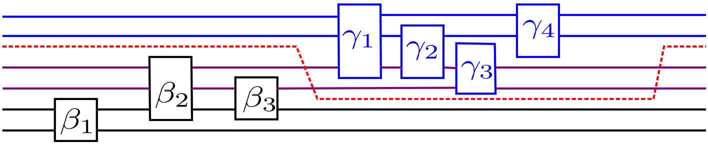
Sketch of an alternative proof (K.):



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Yes (K.), for $K_1 = K_2$ being the 2-knot determined by

$$(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}, w^{-1}\sigma_3 w, w^{-1}\sigma_3^{-1} w), \text{where } w = \sigma_2^2\sigma_1^2\sigma_2^2\sigma_3^2$$

Theorem (K. *2019*)

There cannot be constants C, D so that for any 2-knots K_1, K_2, and any non-trivial 2-knot K, satisfies:

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Thank You!

Ribbon homology cobordisms

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David Shea Vela-Vick[3] *C.-M. Michael Wong[3]

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Washington University in St. Louis

[2]Department of Mathematics
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[3]Department of Mathematics
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Tech Topology Conference 2019

Ribbon cobordisms

- For compact 3-manifolds Y_- and Y_+ (with same ∂), a *cobordism*

$$W : Y_- \to Y_+$$

  is made up of 1-, 2-, and 3-handles
- *Ribbon*: does not have 3-handles
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Why "ribbon"?

- Ans: Related to *ribbon concordances* of knots in S^3, which are concordances with 0- and 1-handles, but no 2-handles

Observation

If $C : K_- \to K_+$ is a ribbon concordance, then the exterior

- $Y_\pm := S^3 \setminus K_\pm$
- $W := (S^3 \times [0,1]) \setminus C$

gives a ribbon cobordism $W : Y_- \to Y_+$.

- Here, homology cobordism means that the maps

$$H_*(Y_-) \to H_*(W) \leftarrow H_*(Y_+)$$

induced by inclusion are isomorphisms.

- W, like C, has no topology in interior (detected by homology)

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Theorem (Gordon 1981)

If $C : K_- \to K_+$ *is a ribbon concordance, then*

$$\pi_1(Y_-) \to \pi_1(W) \leftarrow \pi_1(Y_+).$$

Proof.

Uses the residual finiteness of knot groups $\pi_1(Y_\pm)$. $\square$

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$\pi_1(Y)$ determines the Thurston geometry of Y (if it has one).

Theorem (Daemi–Lidman–Vela-Vick–W.)

If $W : Y_- \to Y_+$ is a ribbon homology cobordism, then

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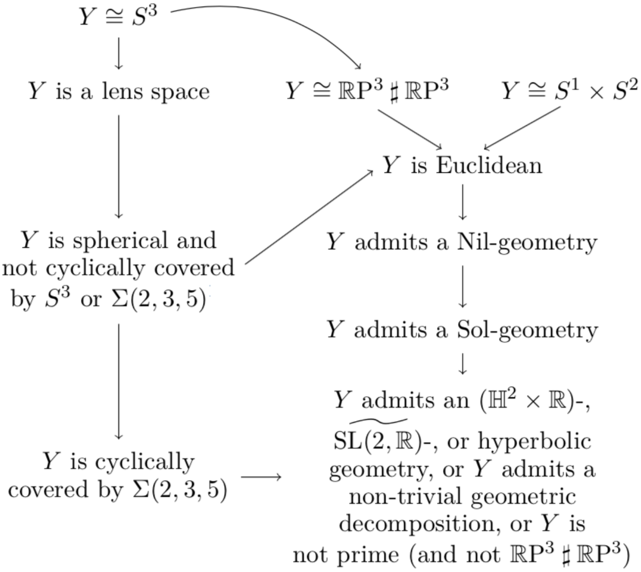

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- Idea: Representations of $\pi_1(Y_\pm)$

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If $W : Y_- \to Y_+$ is a ribbon homology cobordism, then

- *The Thurston geometries of Y_- and Y_+ satisfy a hierarchy.*
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- Any specific G? For example, $\mathrm{SU}(2)$
- Next idea: The $\mathrm{SU}(2)$-representations of $\pi_1(Y)$ are related to the instanton Floer homology $\mathrm{I}^\sharp(Y)$

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- Next idea: Similarly for Heegaard Floer homology!

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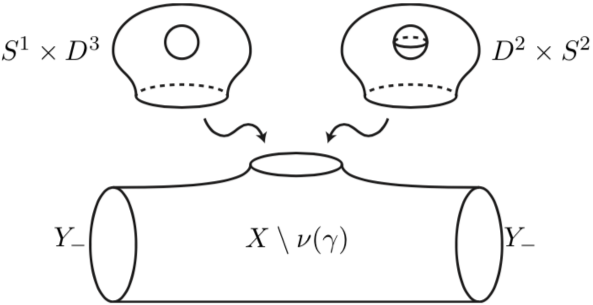
Sketch of proof for Floer homologies

- Doubling trick:



Attaching $S^1 \times D^3 \quad \rightsquigarrow \quad X := (Y_- \times [0,1]) \sharp (S^1 \times S^3)$

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Application to Dehn surgery

Theorem (Daemi–Lidman–Vela-Vick–W.)

Suppose that Y *is a Seifert fibered homology sphere,* K *is a null-homotopic knot in* Y*, and* $Y_0(K) \cong N \,\sharp\, (S^1 \times S^2)$*. Then* $N \cong Y$*.*

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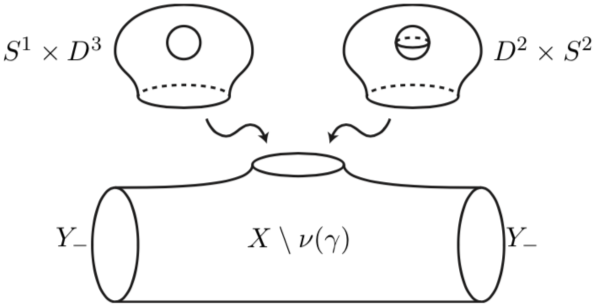

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Finite Quotients of Braid Groups

Lily Li and Caleb Partin
Joint Work with Alice Chudnovsky and Kevin Kordek

Main Result

Finite Quotients of Braid Groups

Let G be a finite group and let $n \geq 5$.

If $B_n \to G$ is not a cyclic homomorphism, then $|G| \geq 2^{\lfloor \frac{n}{2} \rfloor - 1}(\lfloor \frac{n}{2} \rfloor)!$

Definition. Totally Symmetric Set (Kordek, Margalit)
A totally symmetric set X is a subset of a group G which satisfies two properties:

- All elements of the subset commute
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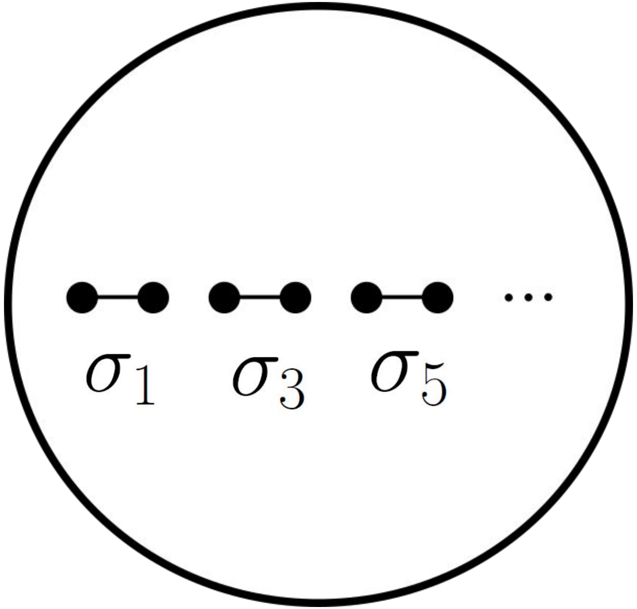

Fundamental Lemma of Totally Symmetric Sets

The image of a totally symmetric set under a homomorphism is a totally symmetric set.

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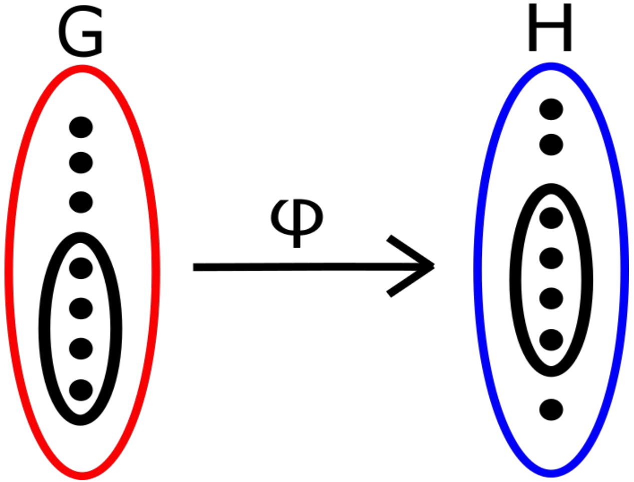
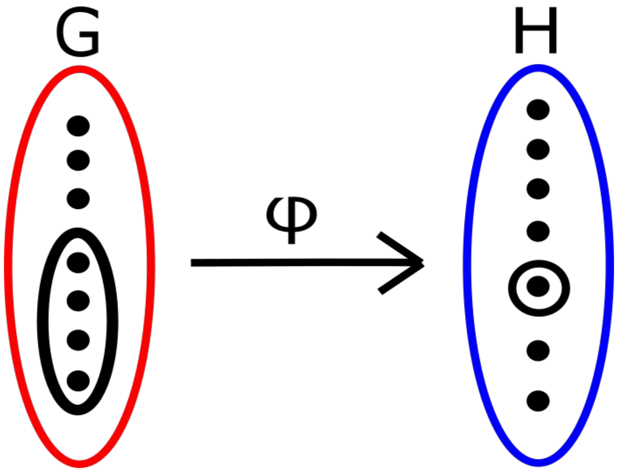
BUT IT GETS BETTER

Fundamental Lemma of Totally Symmetric Sets

The image of a totally symmetric set of size n under a homomorphism is a totally symmetric set of size n or 1.





Proof Outline

$$1 \to K \to \Gamma_S \to S_n \to 1$$

Bounds on Sizes of Totally Symmetric Sets

G	$S(G)$
F_n	1
D_{2n}	2
$\mathbb{Z}/np \rtimes \mathbb{Z}/p$	2
$BS(1,n)$	1 *or* 2
$SL_2(\mathbb{C})$	2

G	$S(G)$
B_n	$\lfloor \frac{n}{2} \rfloor$
S_n	$\geq \lfloor \frac{n}{2} \rfloor$
$Aut(F_n)$	$\geq n$

G	$S(G)$
$G \times H$	$\max(S(G), S(H))$
Ab	1
Odd	1
$Solv$	≤ 4

Constructing free semigroups in nonpositive curvature

Thomas Ng
joint w/
Radhika Gupta and Kasia Jankiewicz

Setting: groups acting on NPC spaces

X: a "nonpositively curved" space

$a, b \in Isom(X)$ (more generally any finite collection)

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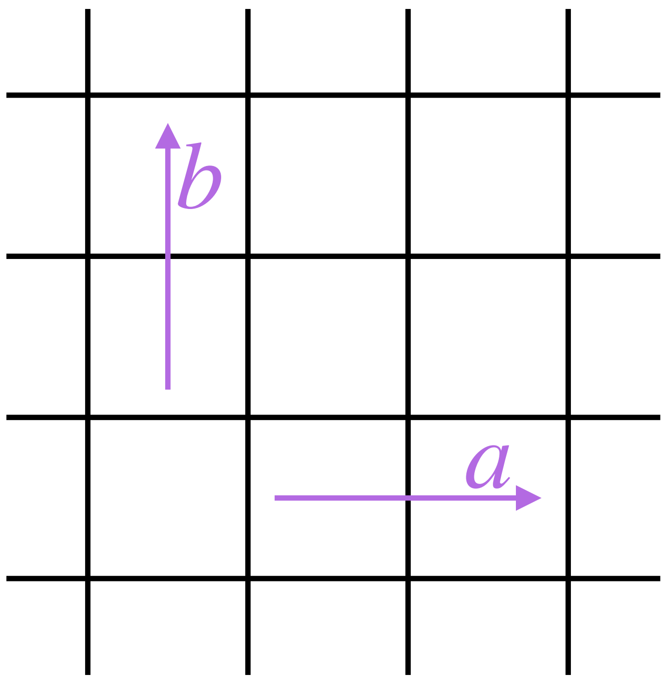

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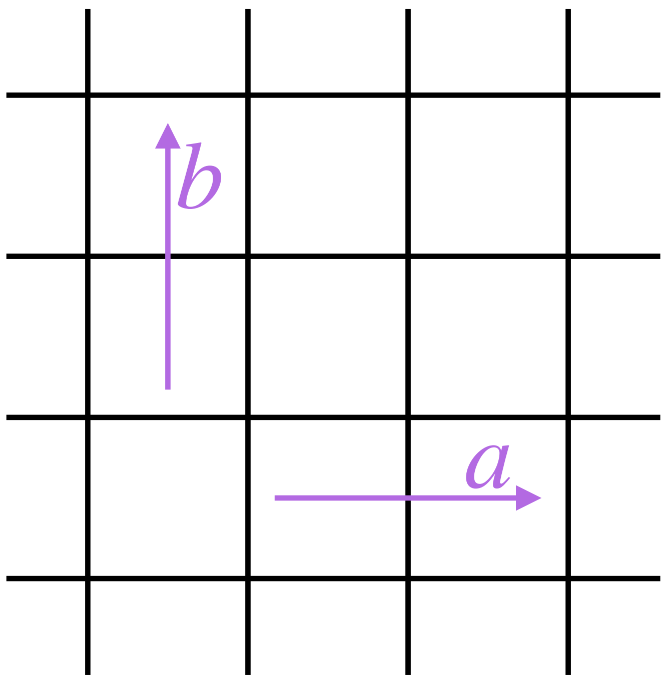

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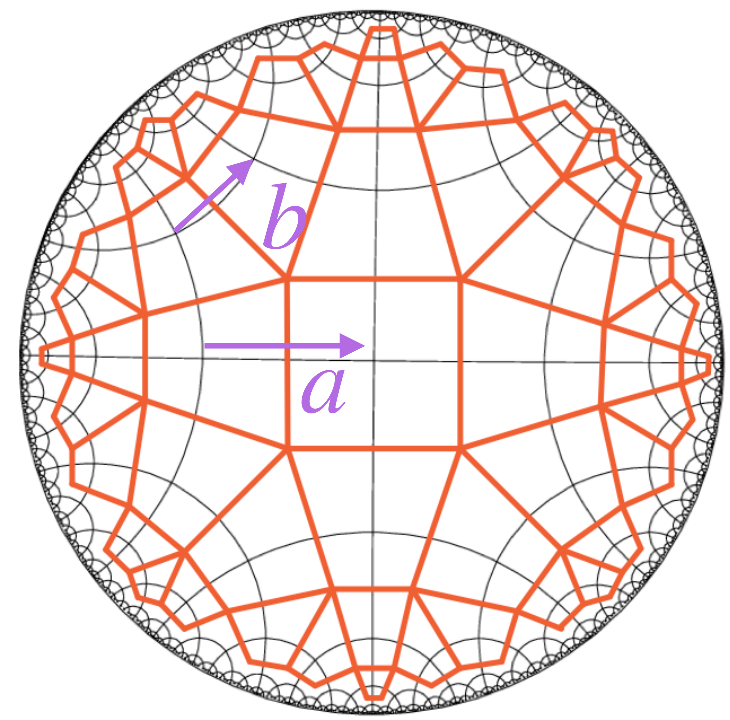

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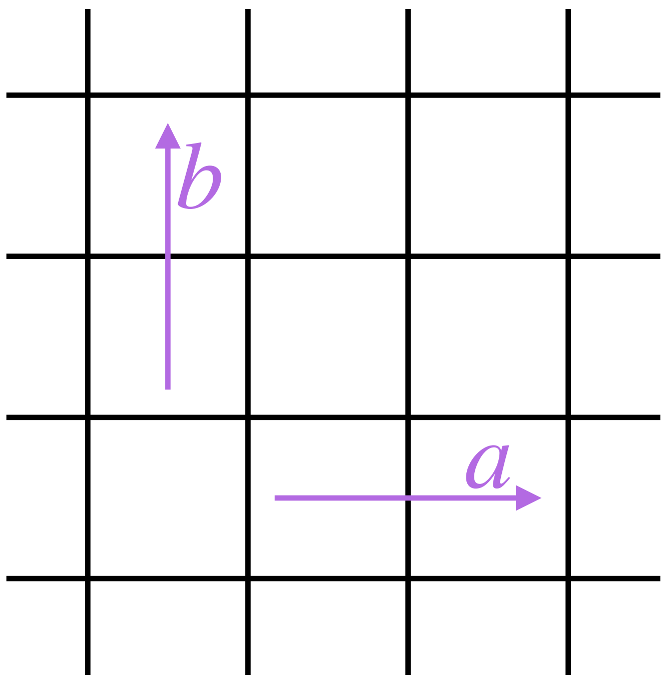

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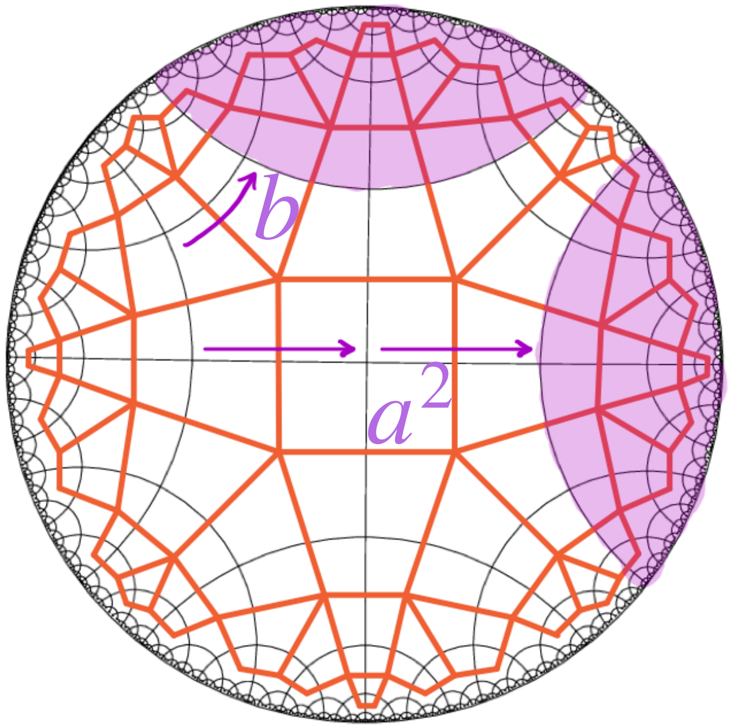

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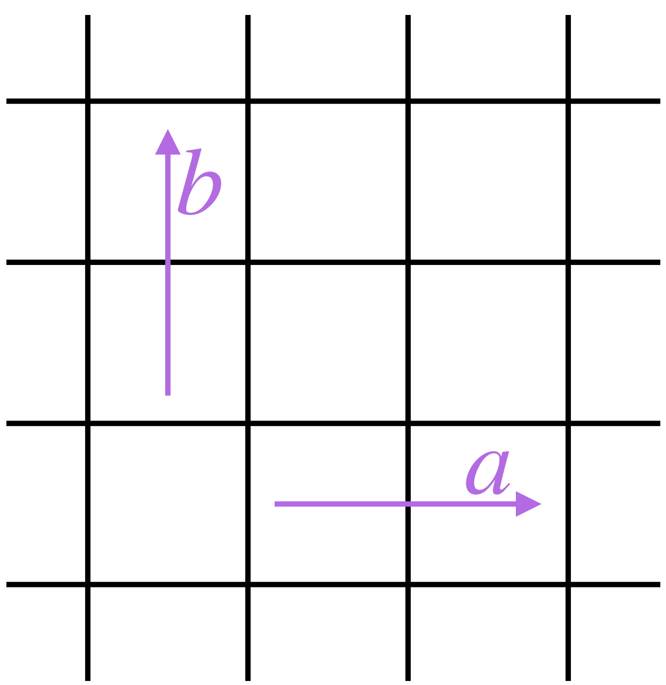

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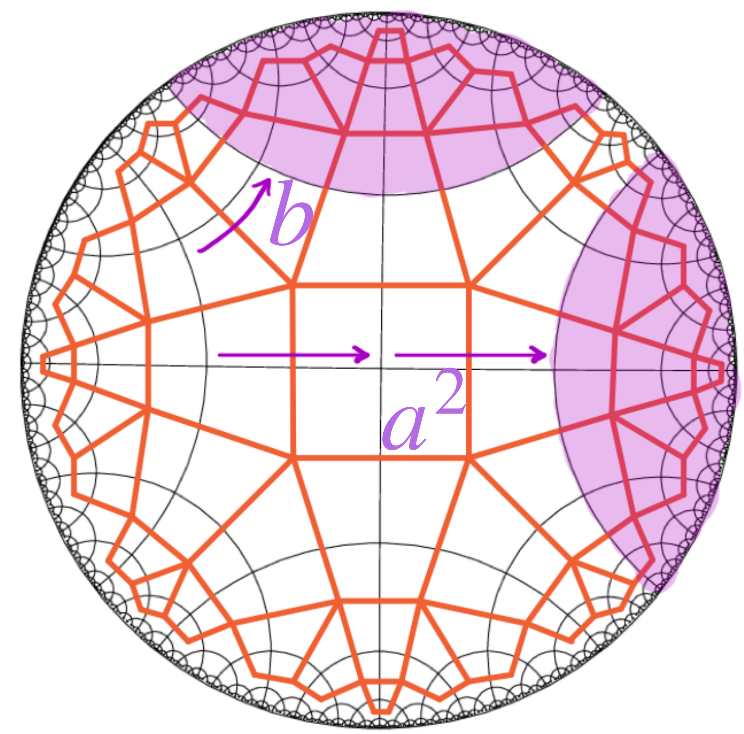

Q: When such a subgroup does exist, can we construct a free basis whose word length is bounded in terms of the geometry of X?

Goal and warm-up

Main result (Gupta, Jankiewicz, Ng)
If X is a CAT(0) cube complex then we can sometimes construct free semigroup bases with length bounded by $dim(X)$.

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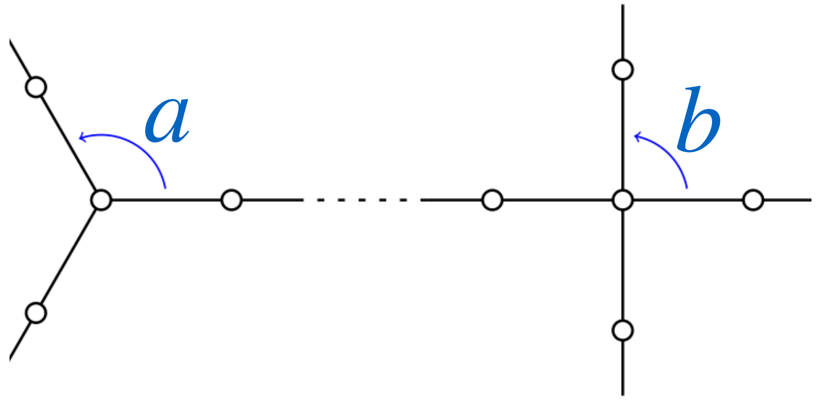

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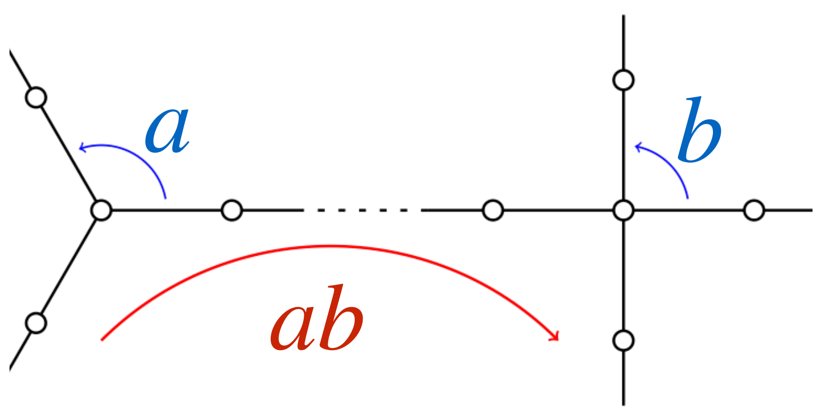

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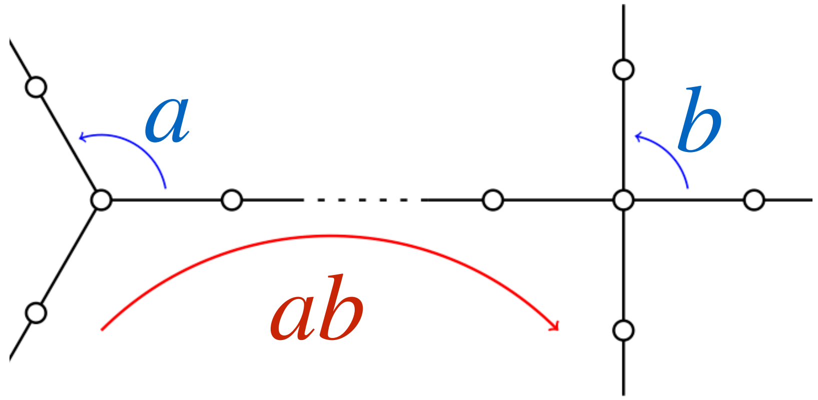

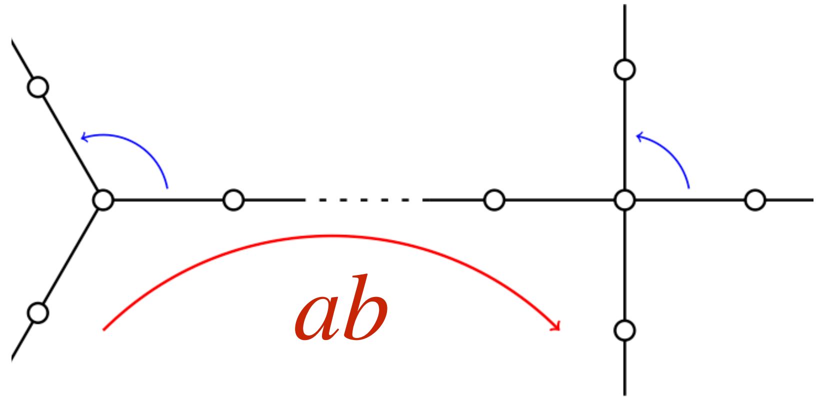

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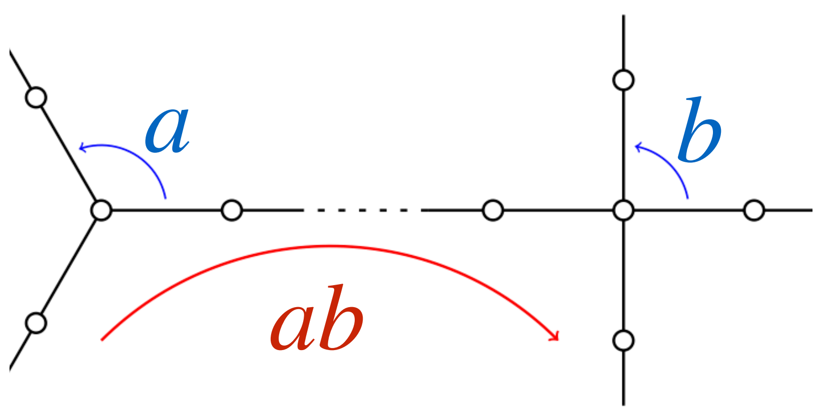

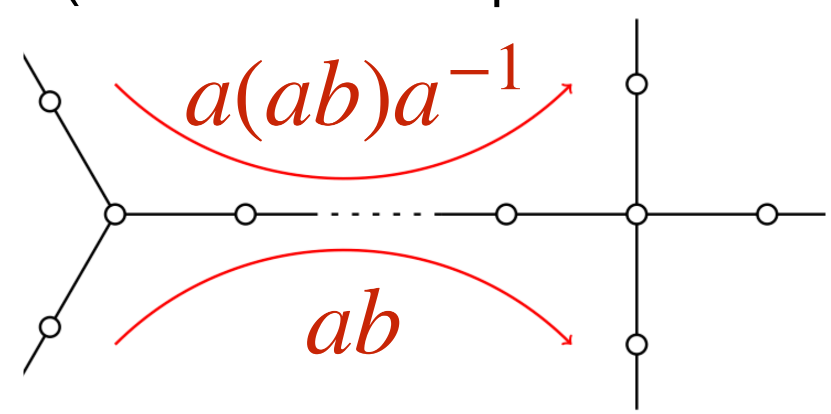

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We want to replicate this for higher dimensional cube complexes

Higher dimensions

Lemma (Gupta, Jankiewicz, Ng)
If $dim(X) \leq 3$ and $\langle a, b \rangle$ acts without global fixed point on X then $\langle a, b \rangle$ contains a hyperbolic isometry with length bounded in terms of $dim(X)$.

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Theorem (Kar, Sageev)
If $dim(X) \leq 2$ and $a, b \in Isom(X)$ are both hyperbolic isometries then either $\langle a, b \rangle$ is virtually abelian, or
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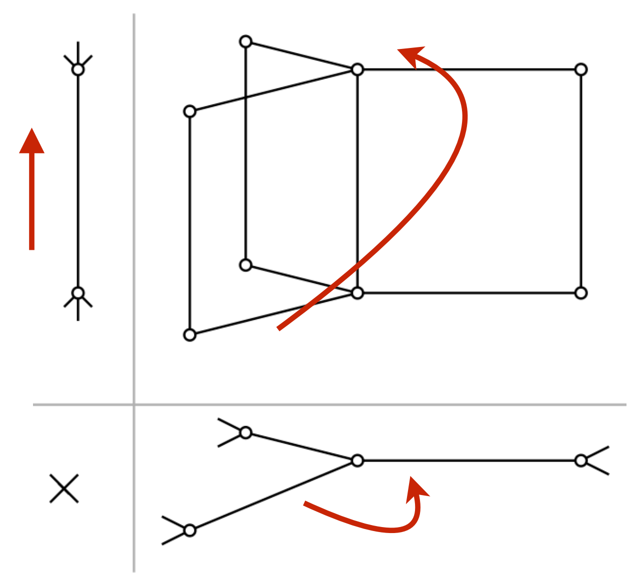
Theorem (Gupta, Jankiewicz, Ng)
If G acts on a finite product of 2-dimensional CAT(0) cube complexes
then either G is virtually abelian,
or G contains a short free semigroup basis.



An application

Certain Artin groups act on 2D CAT(0) cube complexes (Deligne complex) where vertex stabilizers are Artin subgroups.

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If A is a 2D FC-type Artin group and $a, b \in A$ then either $\langle a, b \rangle$ is virtually abelian, or $\langle a, b \rangle$ contains a short free semigroup basis.

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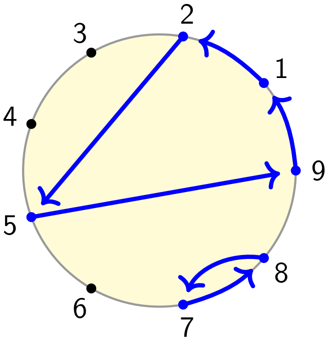
Thanks!

Intrinsic Combinatorics for the Space of Generic Complex Polynomials

Michael Dougherty

December 6, 2019

Colby College

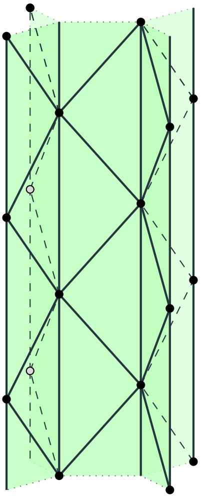


Dual simple braids generate the dual presentation

for $\mathrm{B{\scriptsize RAID}}_n = \pi_1(\mathrm{C{\scriptsize ONF}}_n(\mathbb{C}))$.

dual braid complex: flag complex of Cayley graph

(T. Brady '01, Brady-McCammond '10)



($n = 3$)

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Other $K(\pi, 1)$'s for the pure braid group:

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Q: Can one of these be deformation retracted onto an embedded copy of the quotient complex?

Theorem (D-McCammond, in preparation)

1. $$\left\{ \begin{array}{c} \text{monic degree-}n \\ \text{complex polynomials,} \\ \text{distinct ordered roots} \end{array} \right\} \xrightarrow[\text{retraction}]{\text{deformation}} \left\{ \begin{array}{c} \text{critical values} \\ \text{on unit circle and a} \\ \text{fixed root at zero} \end{array} \right\}$$

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2. This subspace is isometric to the quotient of the dual braid complex by the pure braid group.

Equivalent deformation retraction (June 2019 preprint):

William Thurston, Hyungryul Baik, Yan Gao, John Hubbard, Tan Lei, Kathryn Lindsey, Dylan Thurston.

Homomorphisms between braid groups

Kevin Kordek
joint with Lei Chen and Dan Margalit

Main Theorem (Chen–K–Margalit): A complete classification of all homomorphisms $B_n \to B_m$ with $n \geq 5$ and $n \leq m \leq 2n$.

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Lin, Castel: $B_n \to B_m$ with $n \geq 6$ and $m \leq n + 1$.

Theorem (Chen–K–Margalit, 2019)

For $n \geq 5$ *and* $n \leq m \leq 2n$*, any* $\mathrm{B}_n \to \mathrm{B}_m$ *is equivalent to exactly one of*

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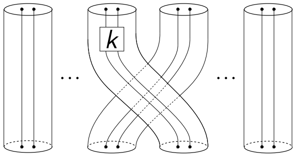
4. *Flip diagonal inclusion*
5. *k-twist cabling*



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- There is only one way to resolve the quartic.

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- This answers a question of Lin from Shpilrain's problem list.

Cabling Legendrian knots

Hyunki Min
with Apratim Chakraborty and John Etnyre

Tech Topology Conference
December 6, 2019

Legendrian knots in $(\mathbb{R}^3, \xi_{std})$

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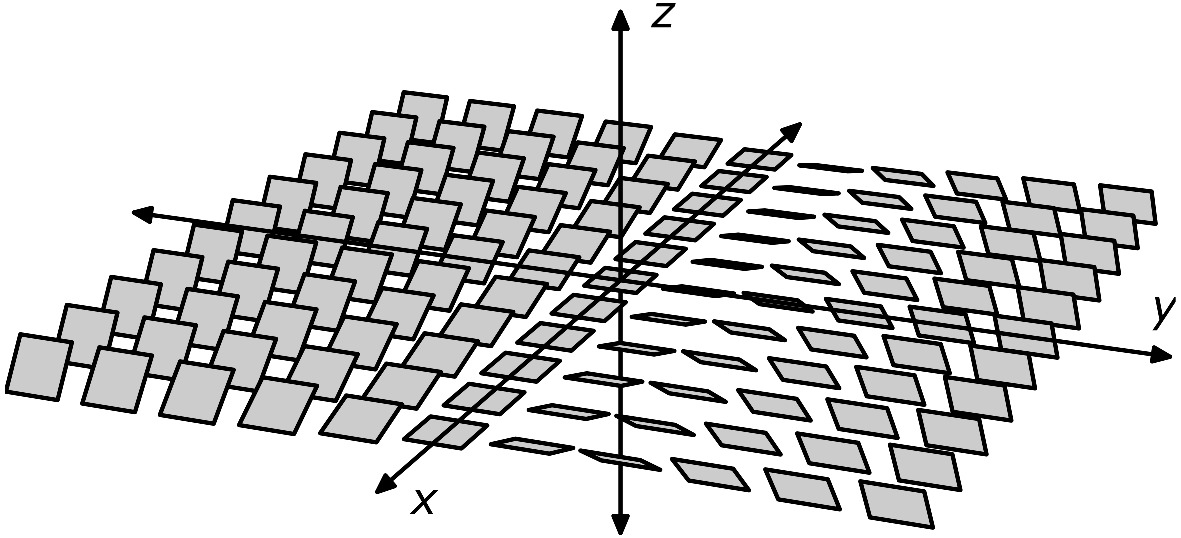

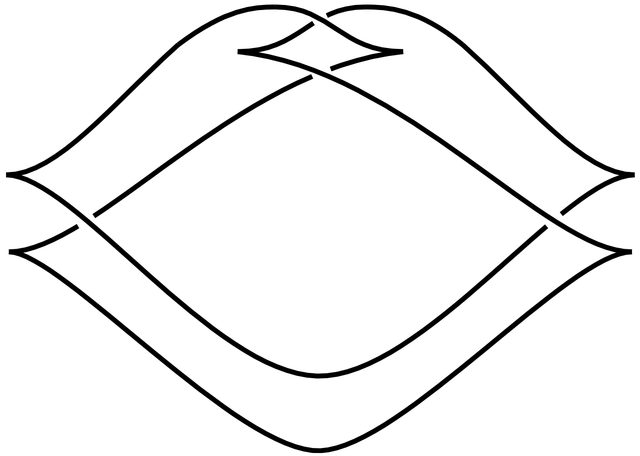

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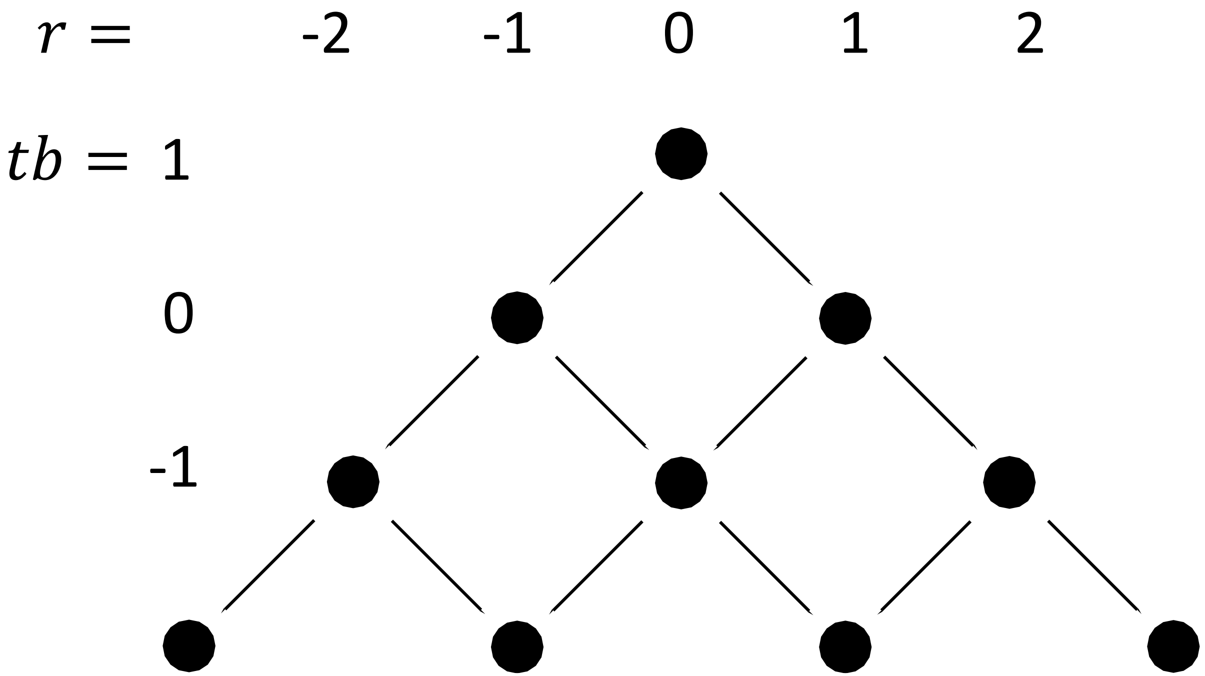
- Rotation number
  A winding number of a tangent field with respect to a trivialization of $\xi|_\Sigma$
- A knot is called Legendrian simple if its Legendrian isotopy classes are determined by Thurston-Bennequin and rotation numbers.

Mountain range



Right-handed trefoil

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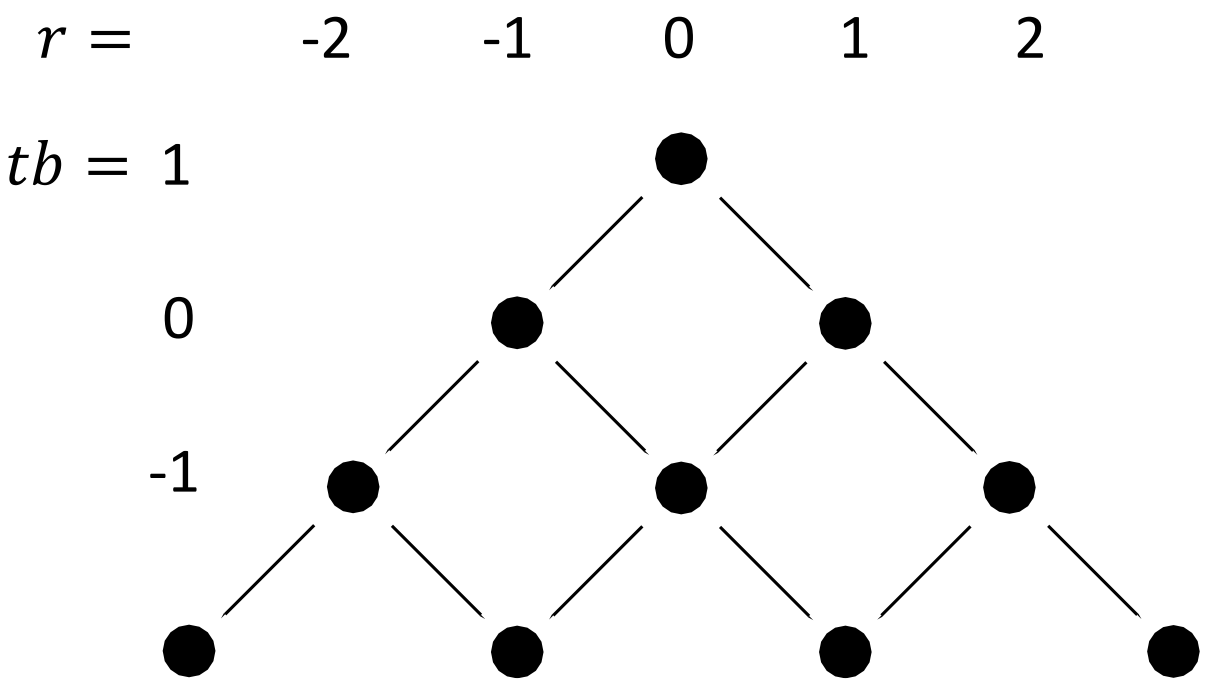

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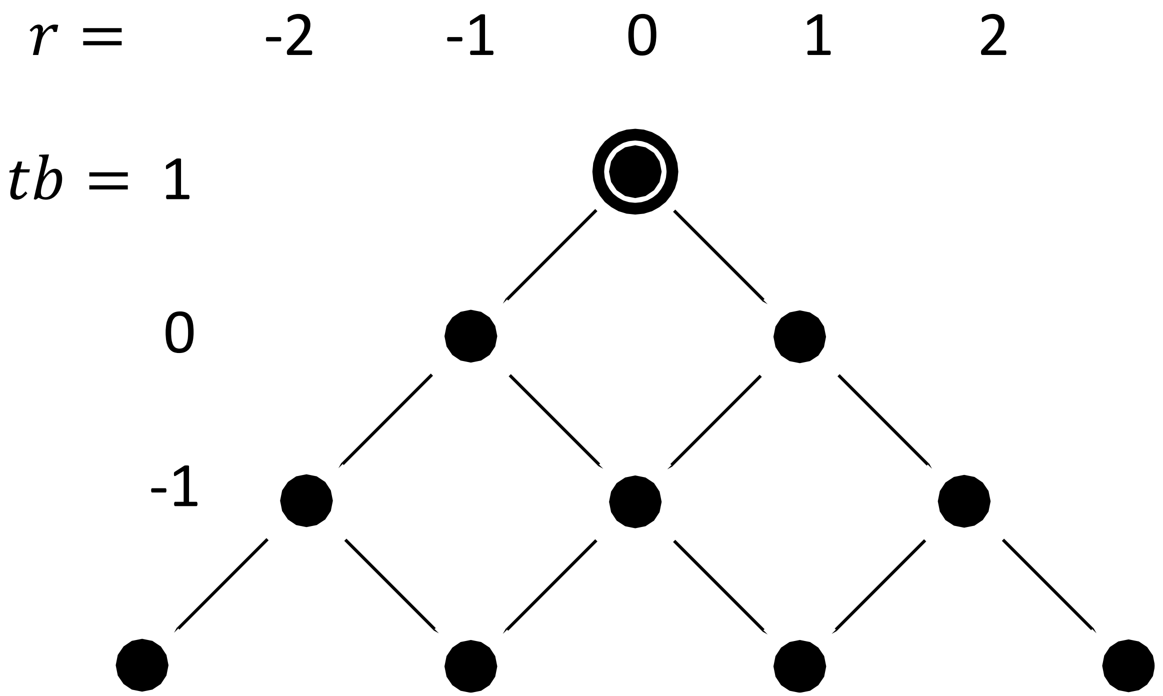

$m5_2$

Legendrian cables

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- (Chakraborty) $\hat{\theta}(T^1) = \hat{\theta}(T^2)$ if and only if $\hat{\theta}\left(T^1_{p,q}\right) = \hat{\theta}\left(T^2_{p,q}\right)$

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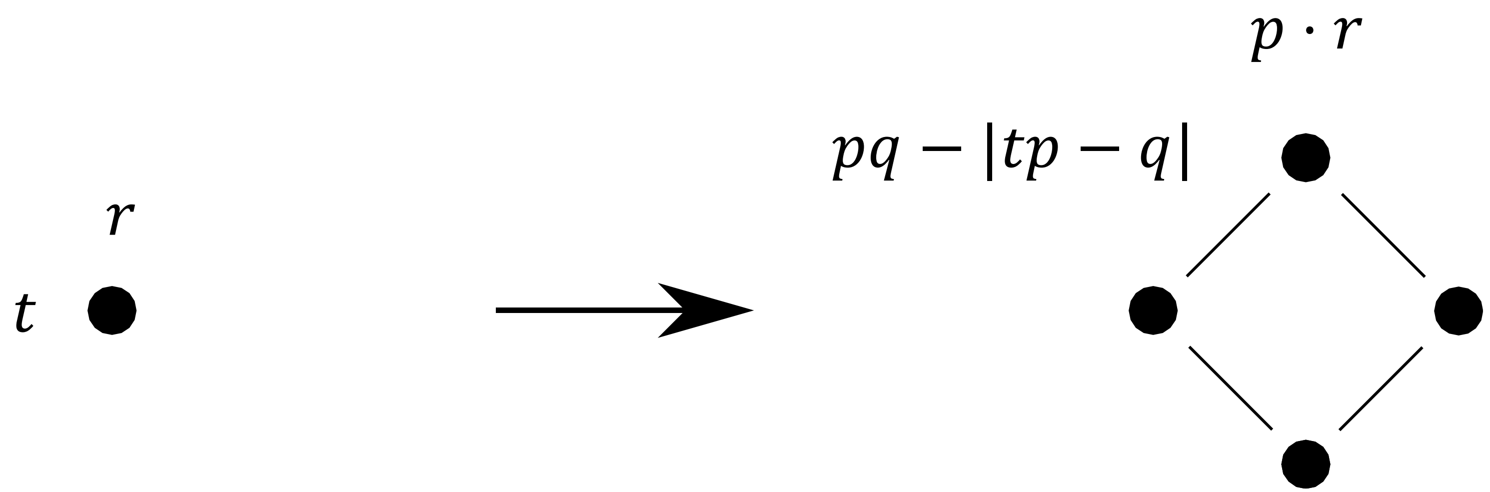
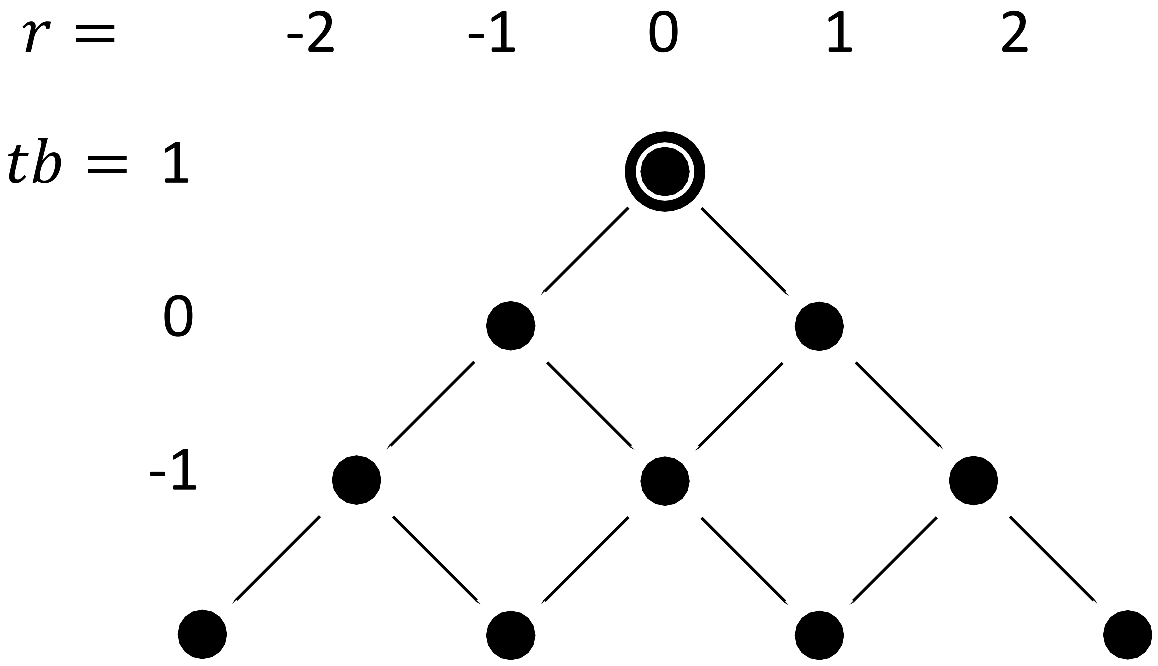
- For $q/p > \overline{tb}(K) + 1$, $K_{p,q}$ is Legendrian simple if and only if K is Legendrian simple.
- The mountain range of $K_{p,q}$ is a (p,q)-diamond of the mountain range of K.

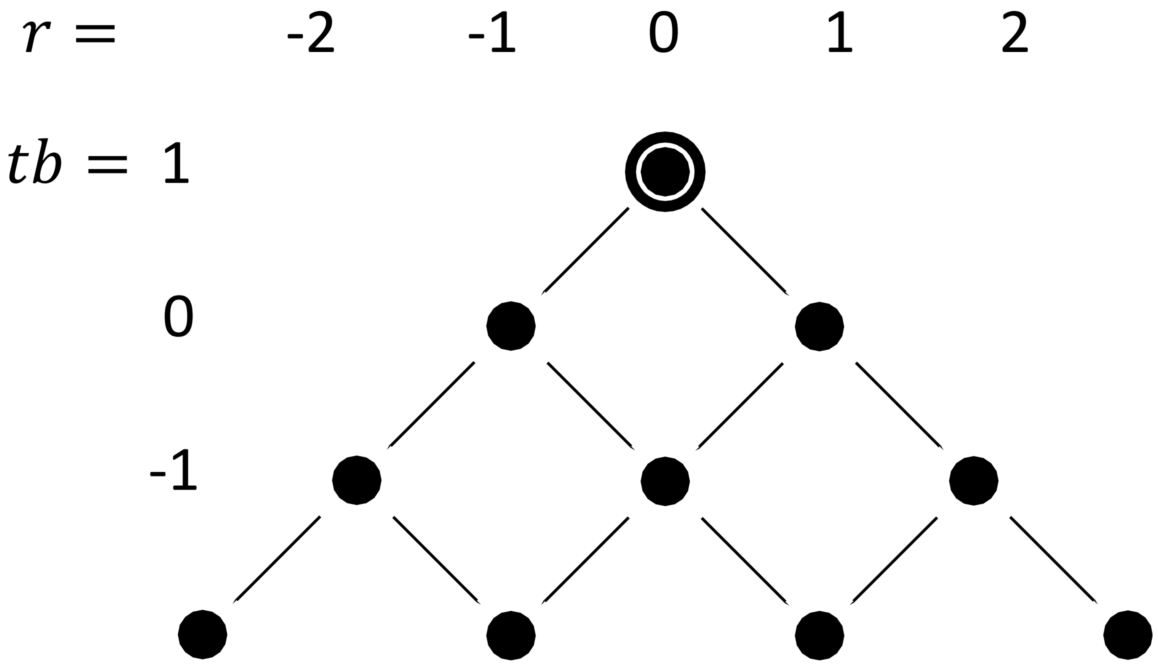


Examples

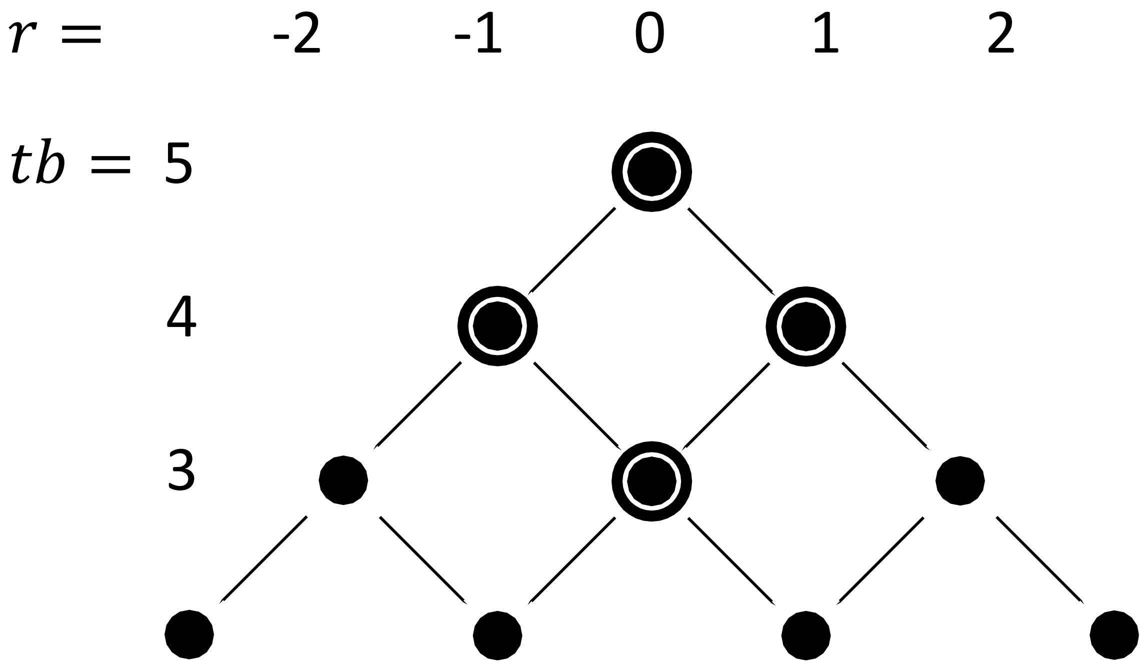


$m5_2$

Examples



$m5_2$



(2,3)-cable of $m5_2$

Idea of Proof

- Put a cable $L_{p,q}$ on a standard neighborhood of L.
- Assume we have a common $L_{p,q}$ on neighborhoods of L and L'

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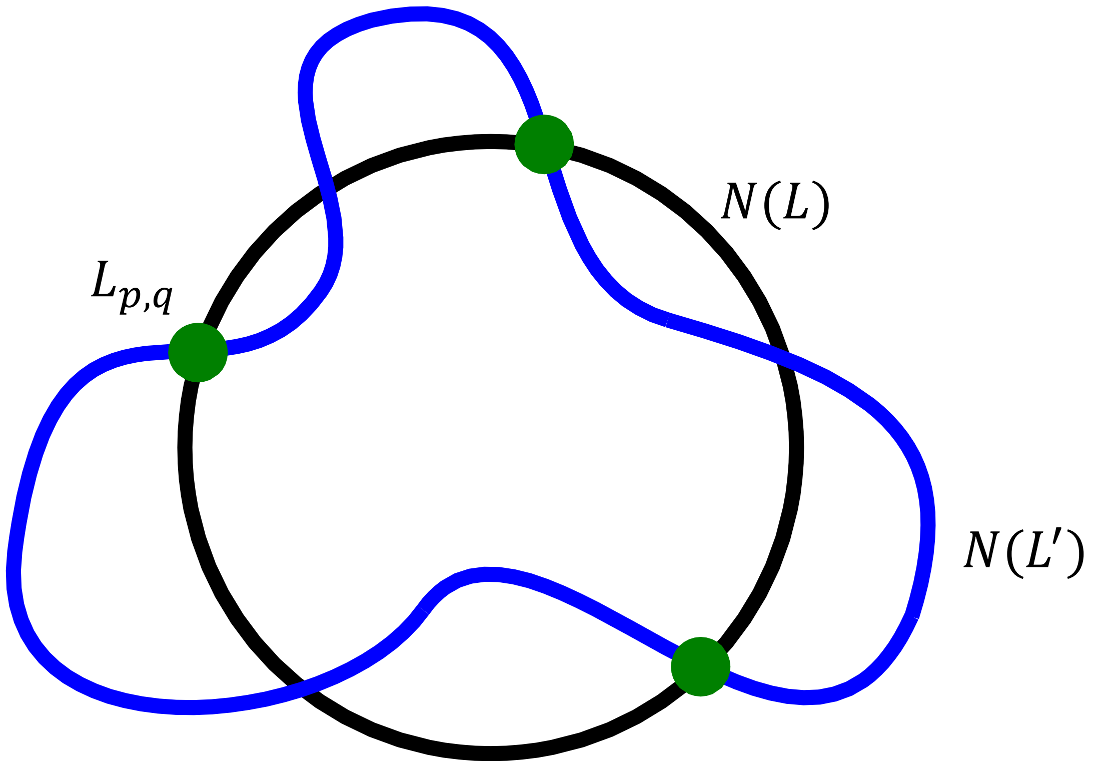
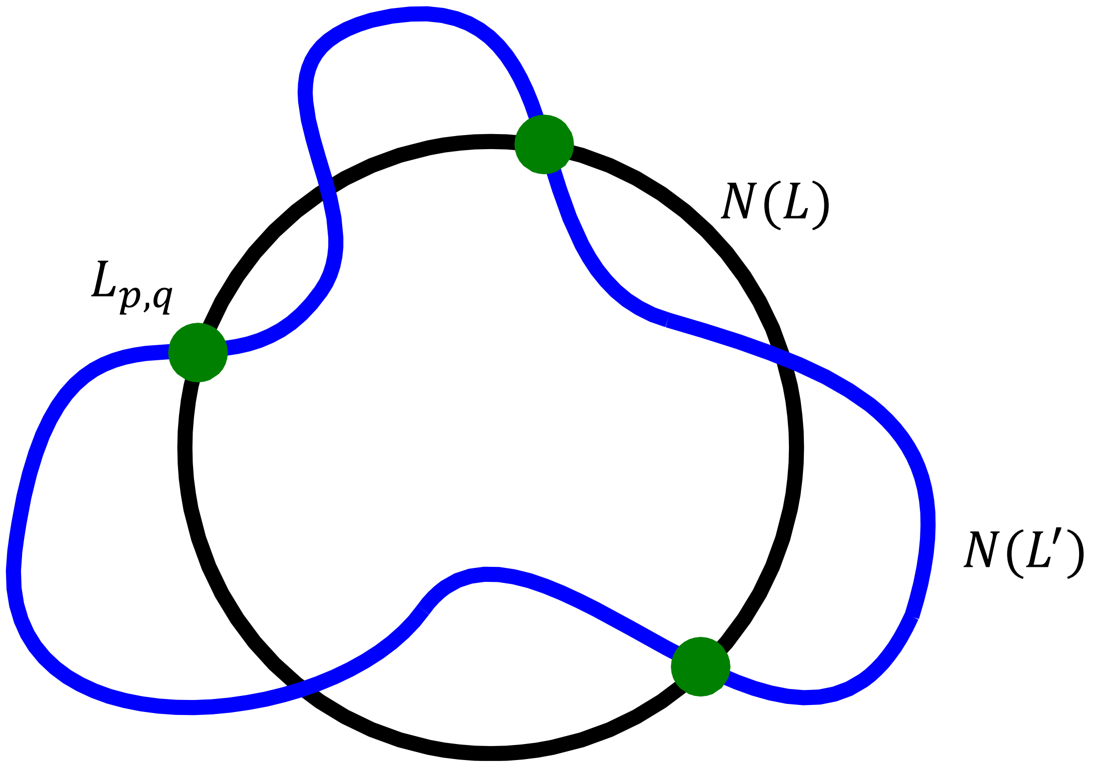
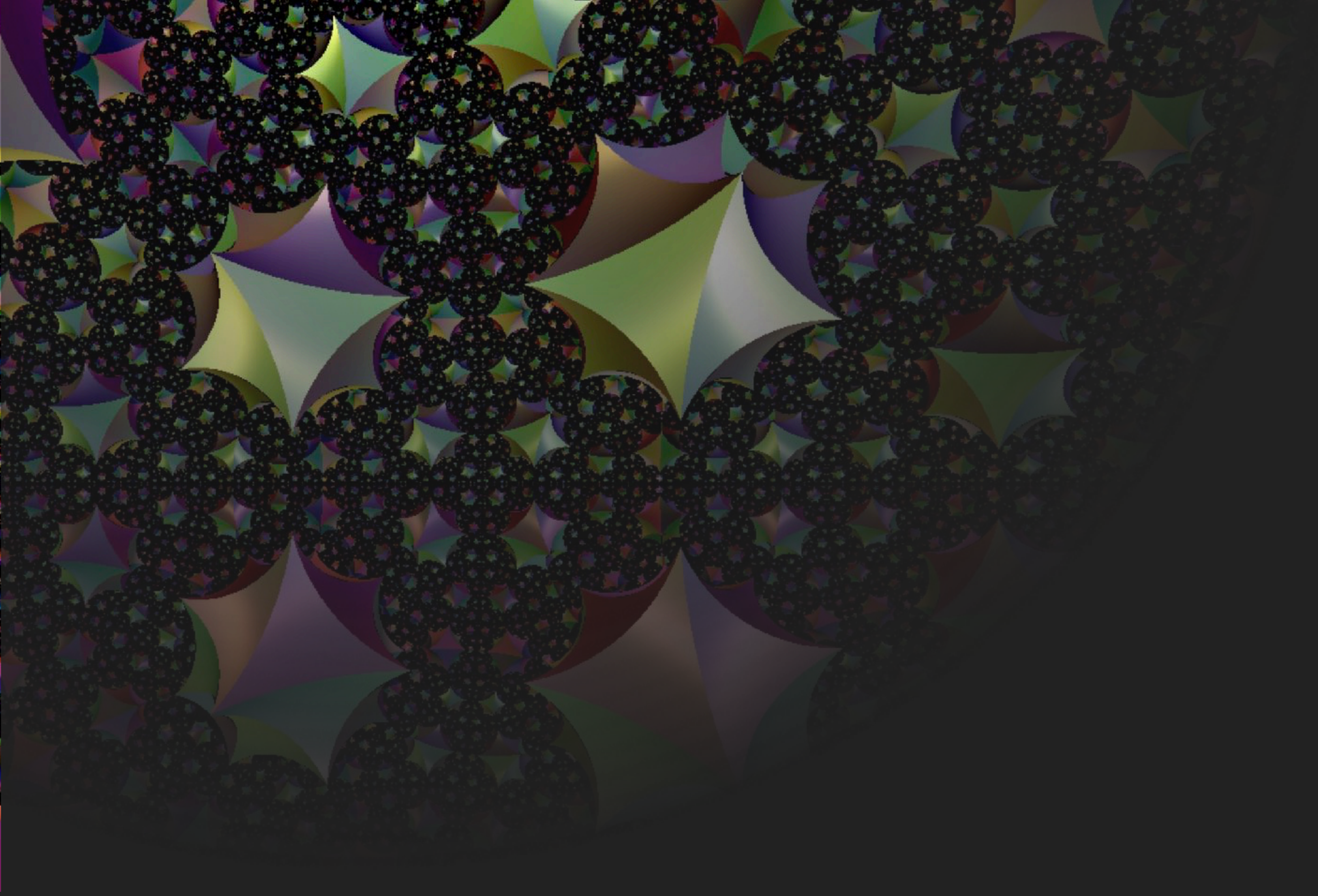


Idea of Proof

- There is a smooth isotopy from $N(L)$ to $N(L')$ fixing $L_{p,q}$
- Keep track of the contact structure on $N(L)$ during the isotopy



Thank you!



STEVE TRETTEL
ICERM

RAYMARCHING

HOMOGENEOUS GEOMETRIES

JOINT WITH

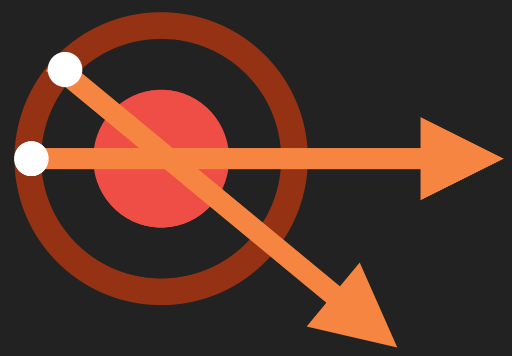
Henry Segerman

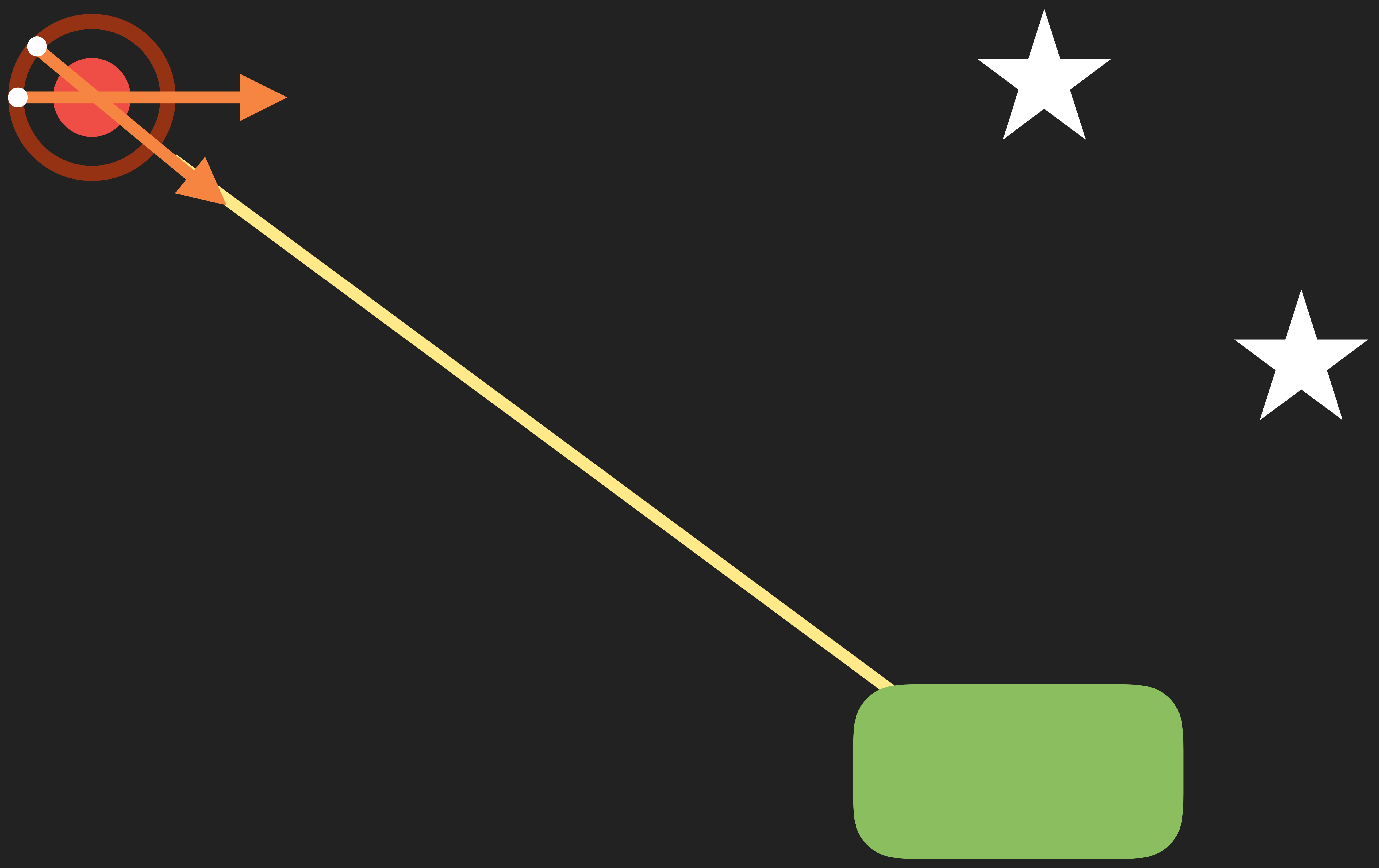
Sabetta Matsumoto

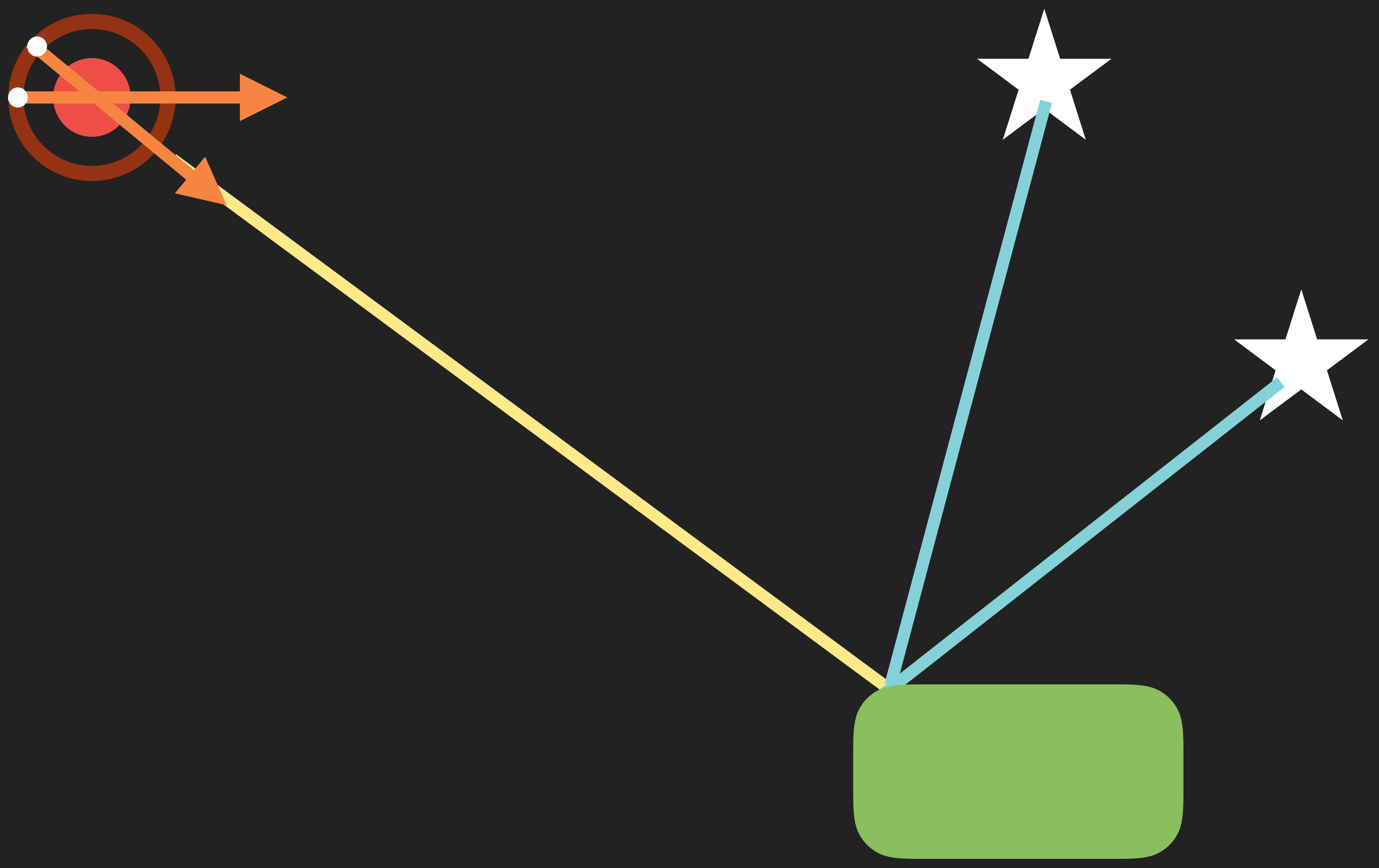
Remi Coulon

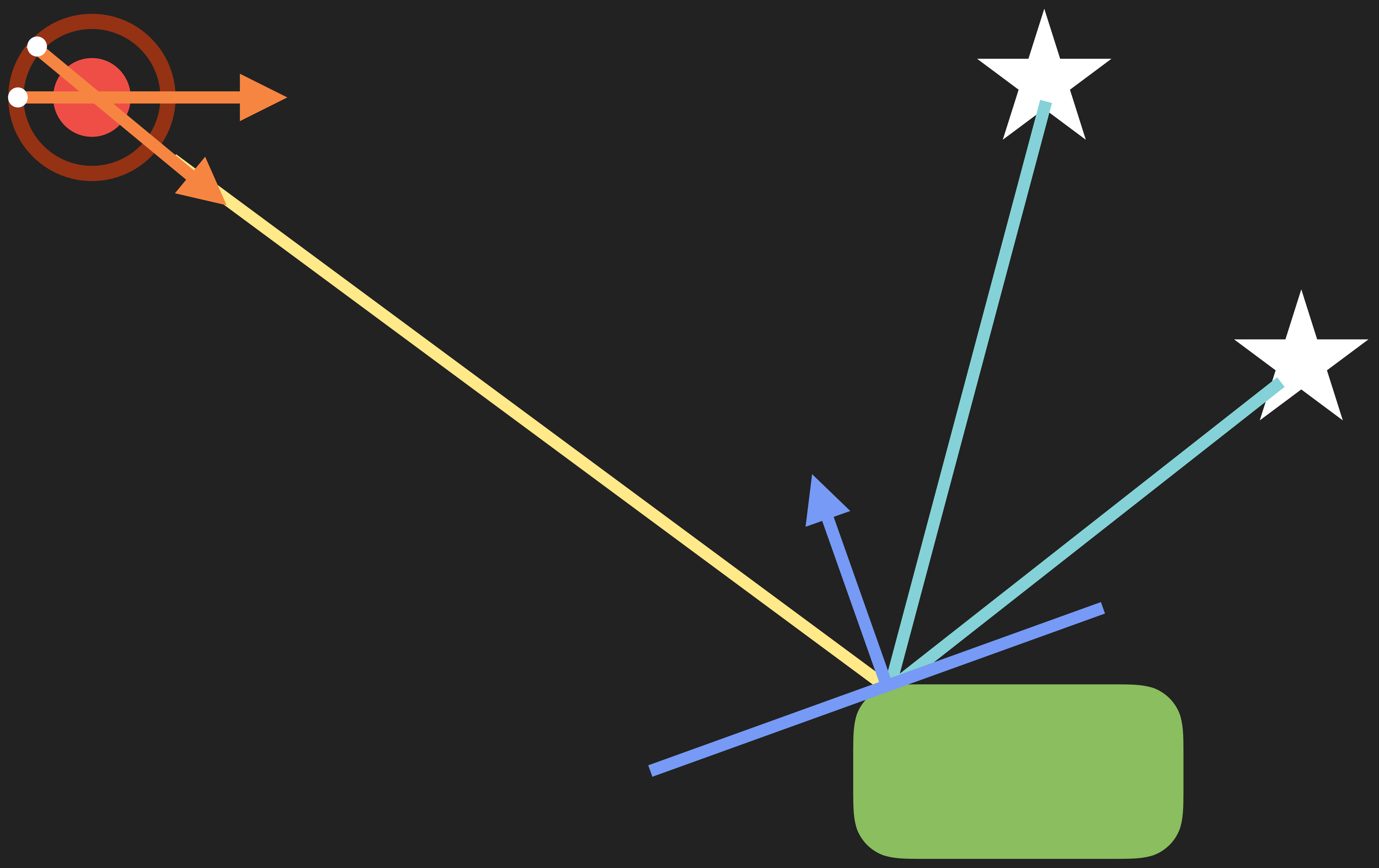
Brian Day

WHAT DOES IT MEAN TO SEE ?









Begin at a point $p \in M$

Pixels are points of T^1_pM

From $v \in T^1_pM$, follow $\gamma_v : \mathbb{R}_{\geq 0} \to M$

When γ_v intersects X at q, stop.

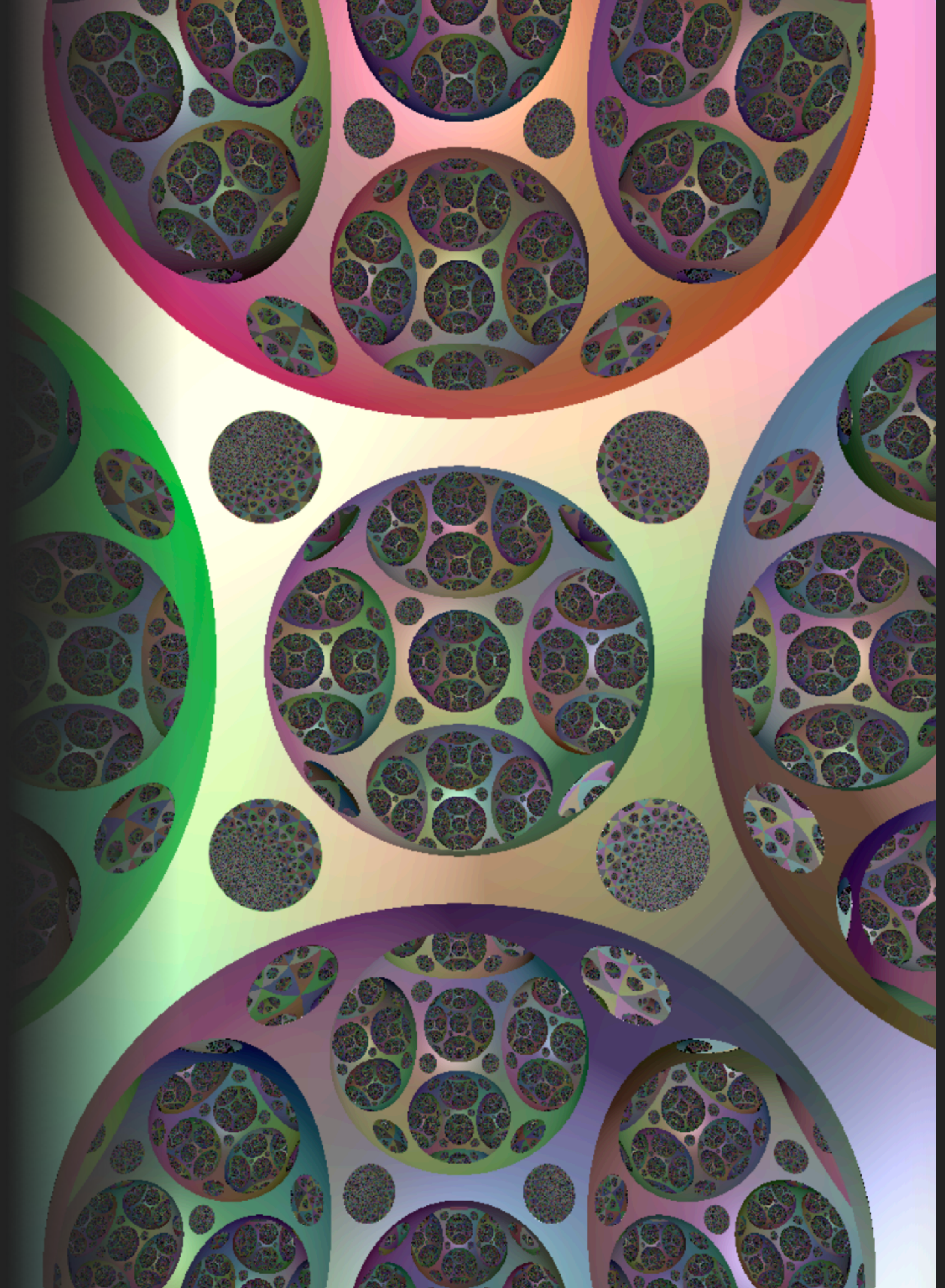
Find directions $\{v_i\} \in T^1_q\mathbb{E}^3$ to lights.

Compute surface normal $n_q \perp T_qX$

Normal + lighting $\longrightarrow$ pixel color

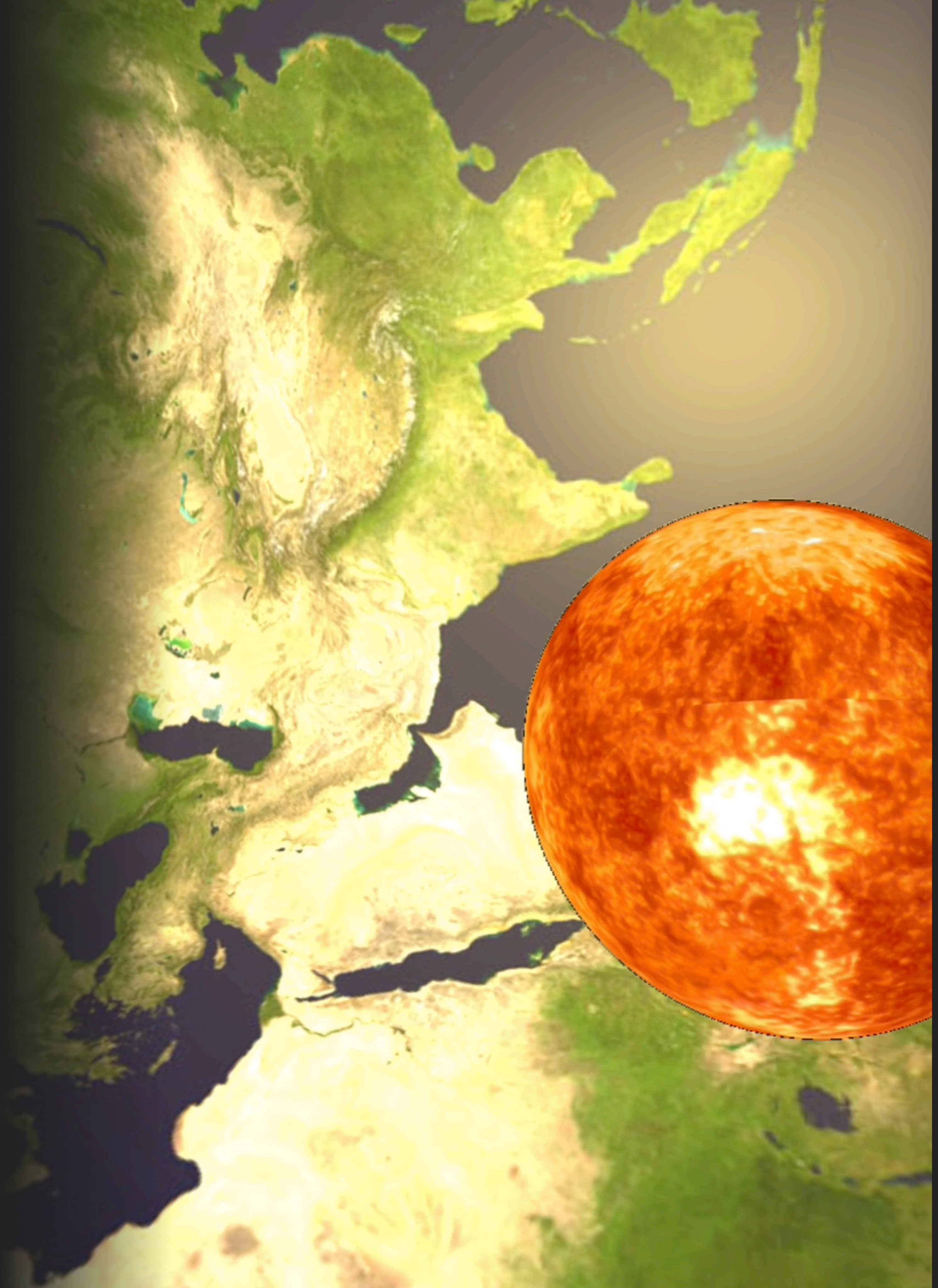
$\mathbb{H}^3$

FLY

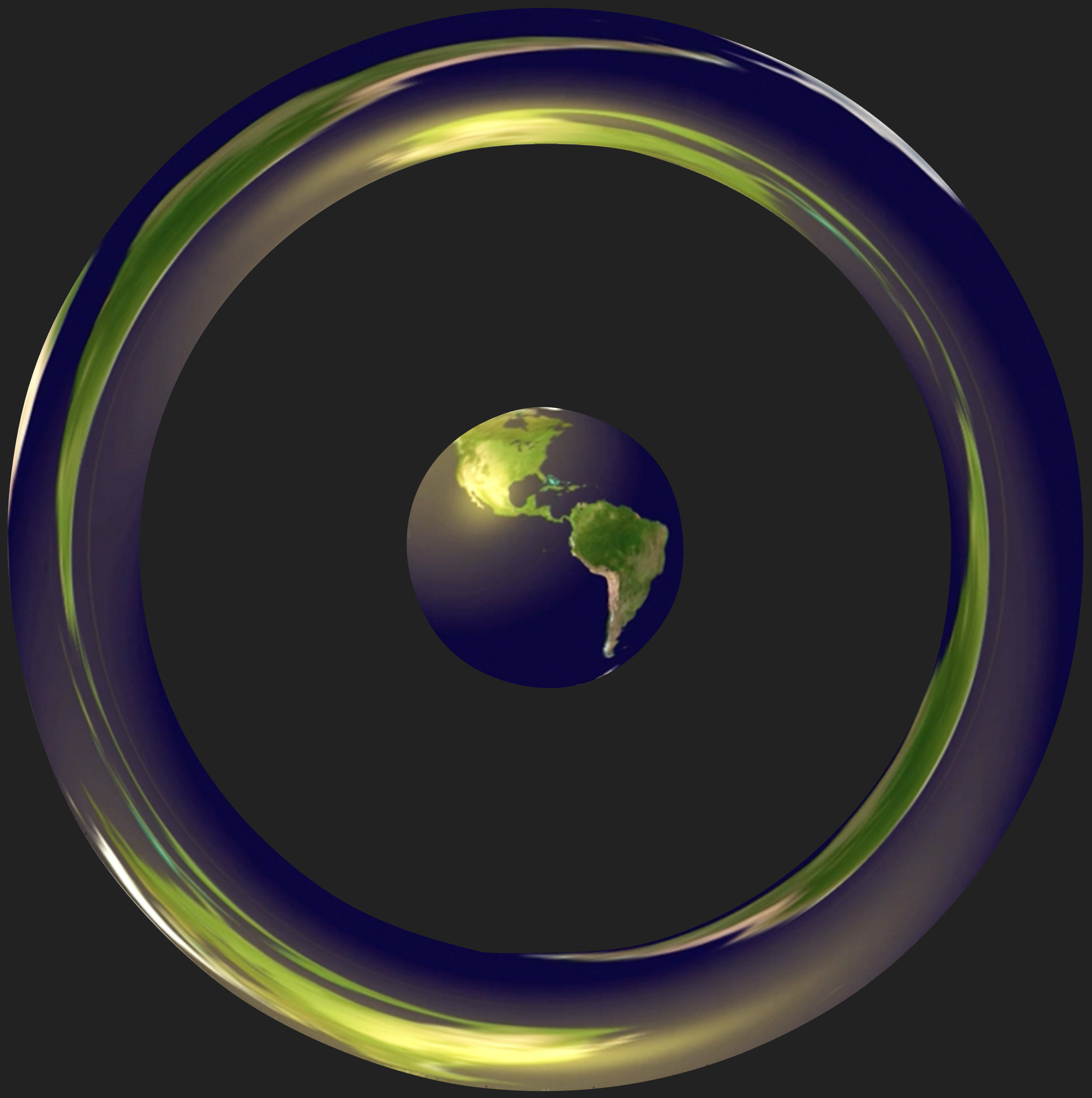


$\mathbb{S}^3$

FLY



Nil



FLY