Lightning Talks II
Tech Topology Conference

December 7, 2019

Constraining mapping class group homomorphisms using finite subgroups

Justin Lanier, Georgia Tech

with Lei Chen

Conjecture (Mirzakhani)

MCG homomorphisms

finite image

"induced by
some manipulation
of surfaces"

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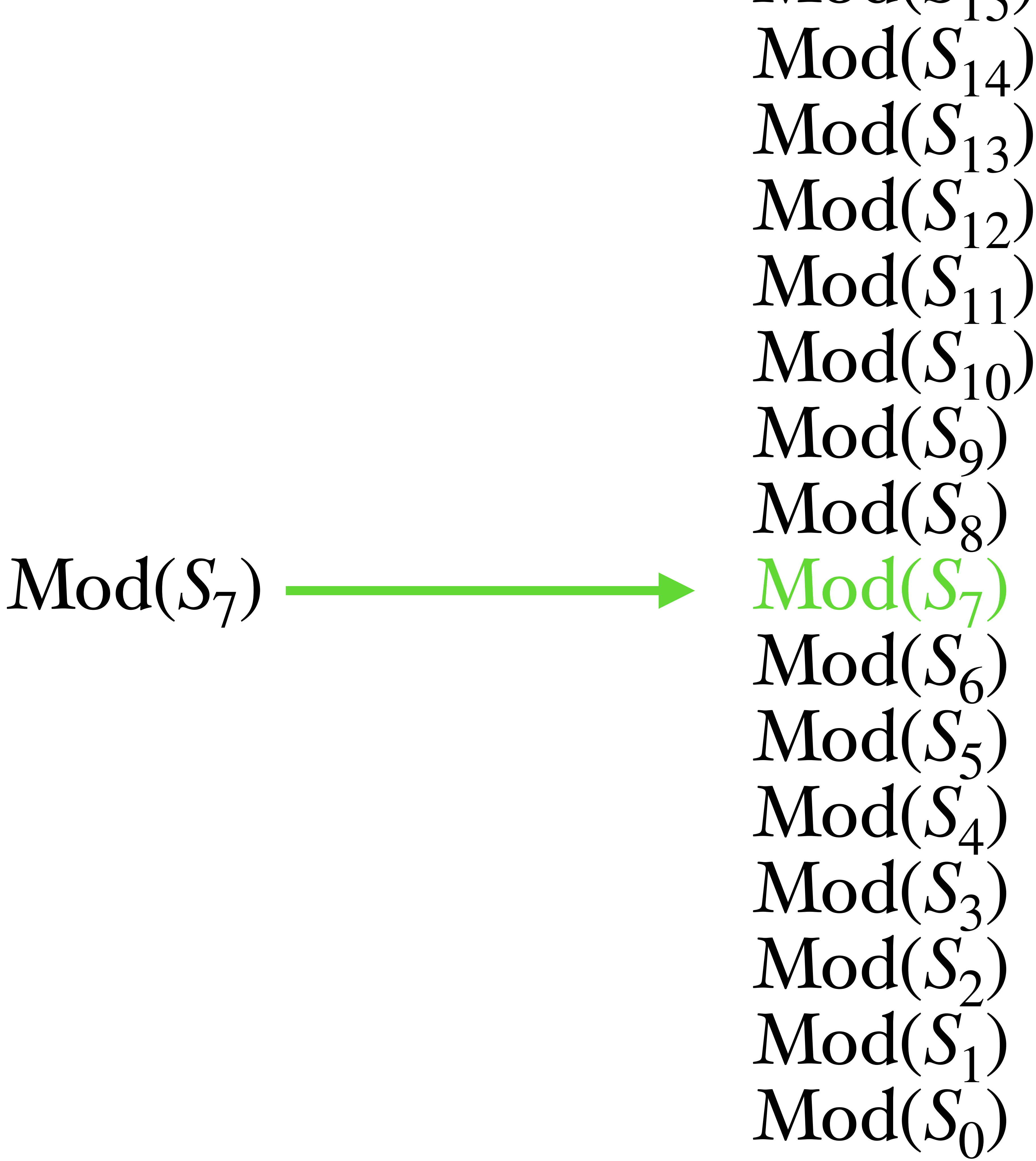

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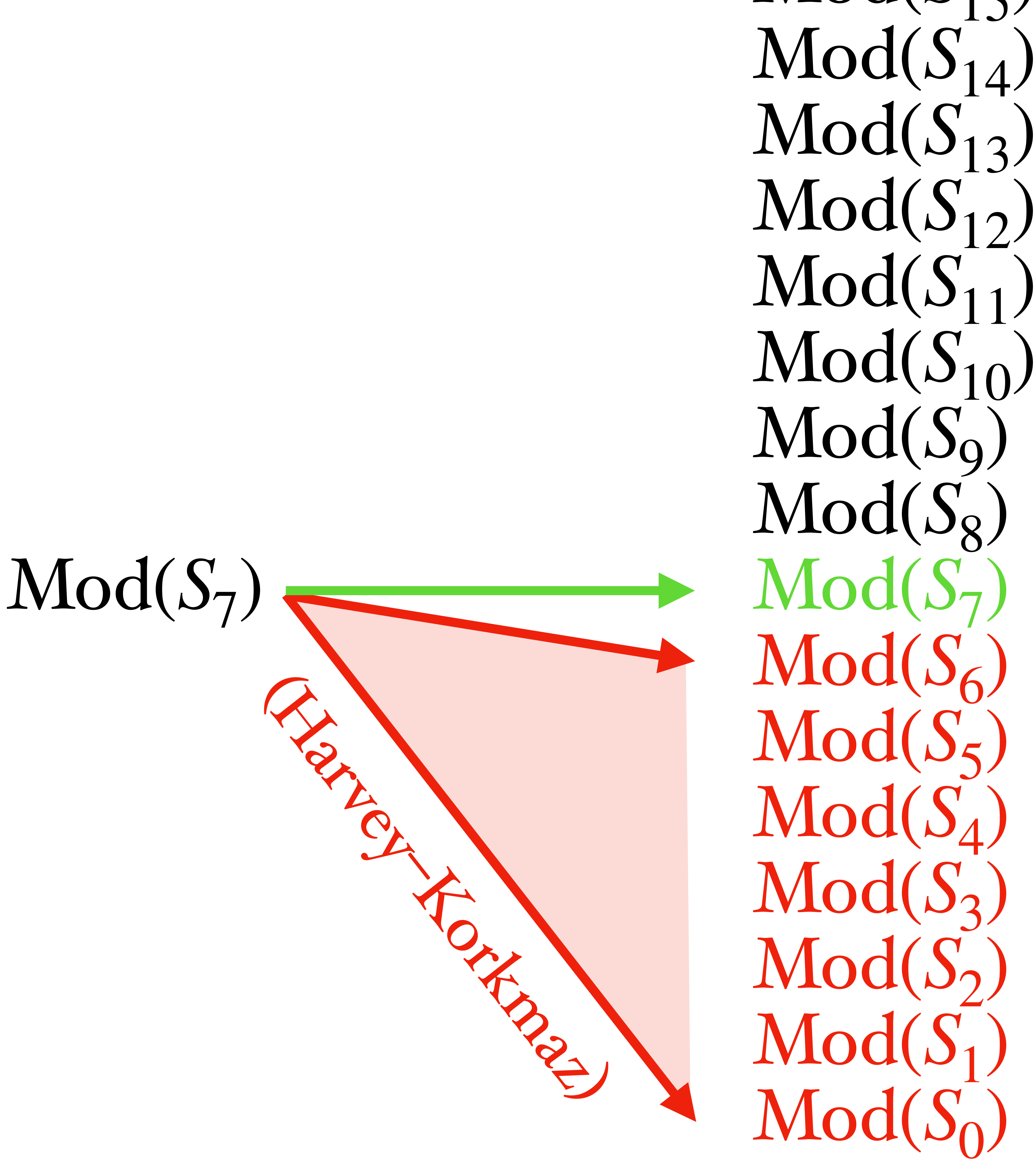

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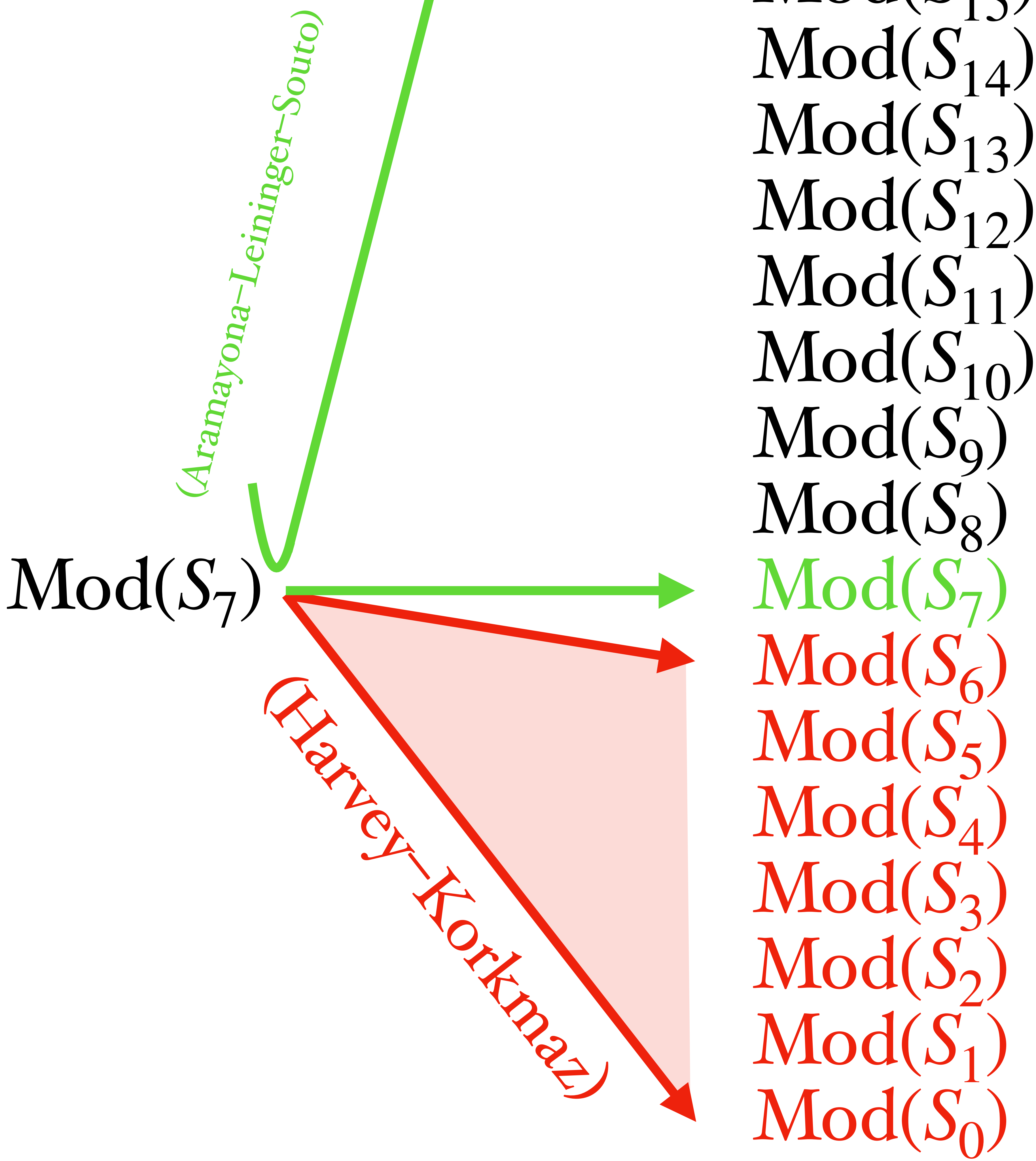
(Harvey–Korkmaz)



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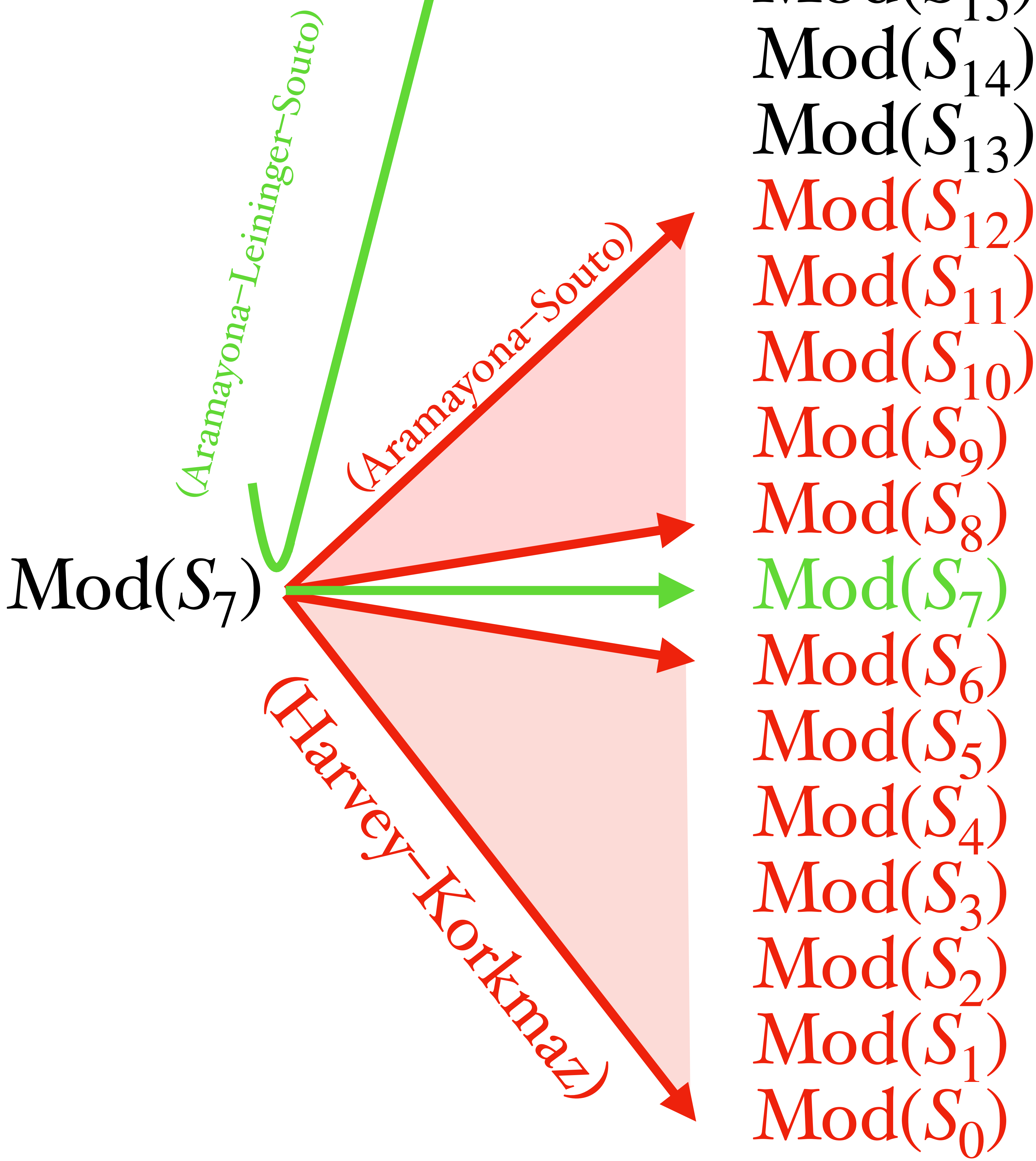
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Theorem (Aramayona–Souto)

For $g \geq 6$ and $g' < 2g - 1$, every nontrivial homomorphism $\mathrm{Mod}(S_{g,n,b}) \to \mathrm{Mod}(S_{g',n,',b'})$ is induced by an embedding.

So for closed surfaces, isomorphism or trivial.

Proof (Aramayona–Souto)

Dehn twists go to roots of multitwists

(Bridson)

⇝

Dehn twists go to Dehn twists

⇝

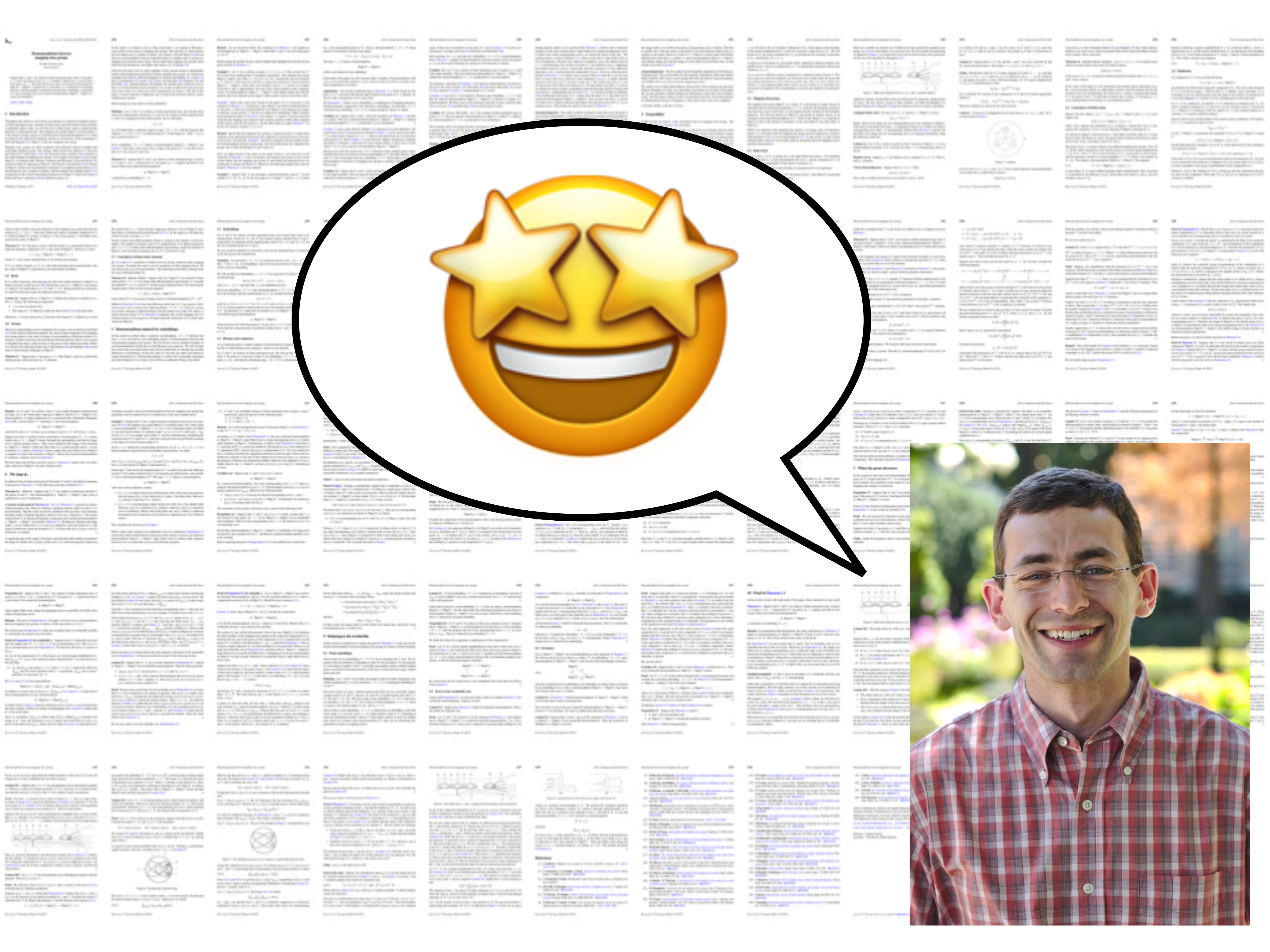
A chain of curves, and an embedding

Homomorphisms between mapping class groups

[illegible]

1 Introduction

[illegible]



Proof (Chen–L)

torsion dies $\Longrightarrow$ homomorphism trivial	+	torsion dies	$\Longrightarrow$	homomorphism trivial

Proof (Chen–L)

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Theorem (L–Margalit)

For $g \geq 3$, every nontrivial periodic mapping class that is not a hyperelliptic involution normally generates $\mathrm{Mod}(S_g)$.

Proof (Chen–L)

torsion dies $\implies$ homomorphism trivial	+	torsion dies	$\implies$	homomorphism trivial

Theorem (L–Margalit)

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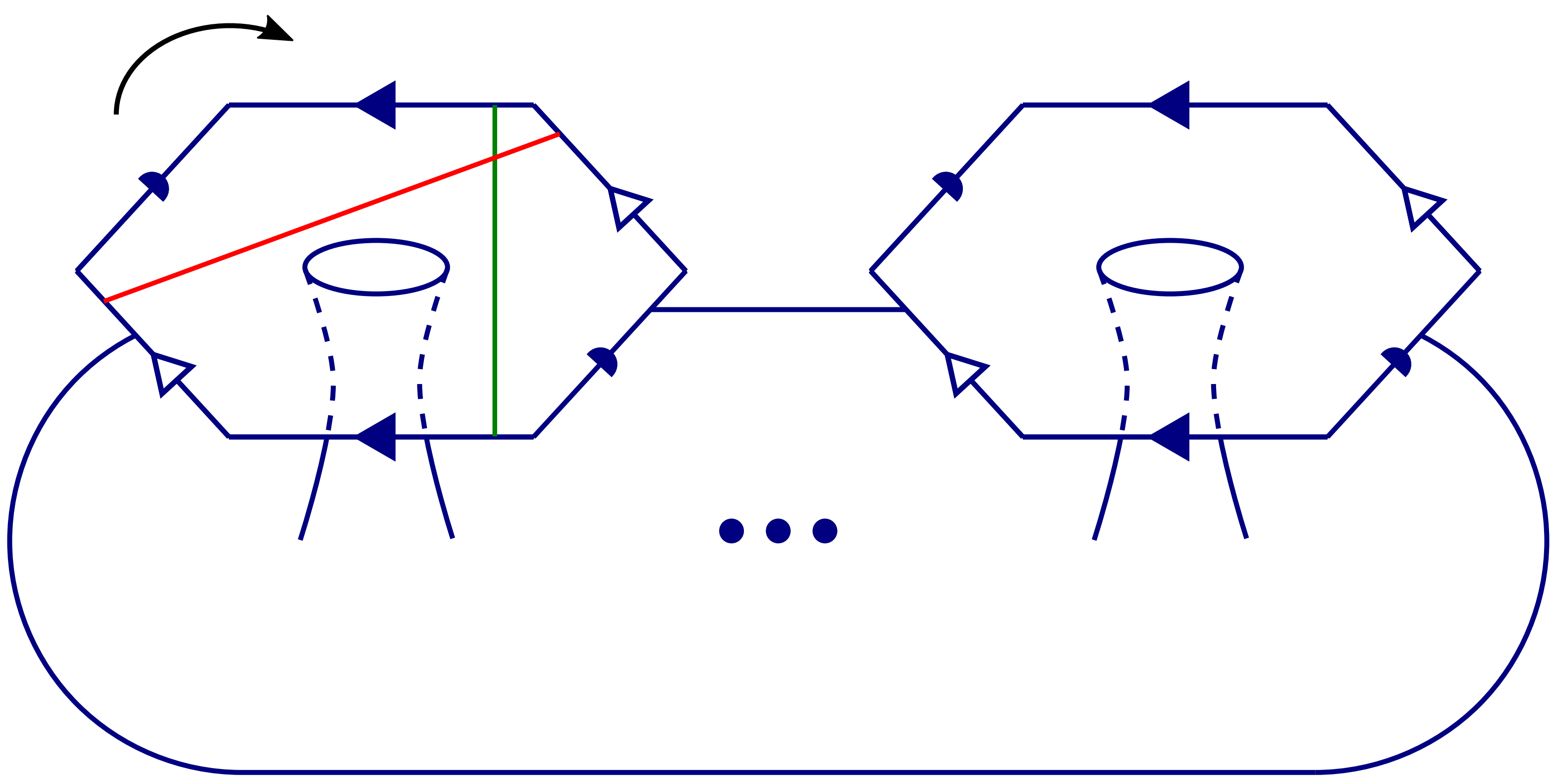

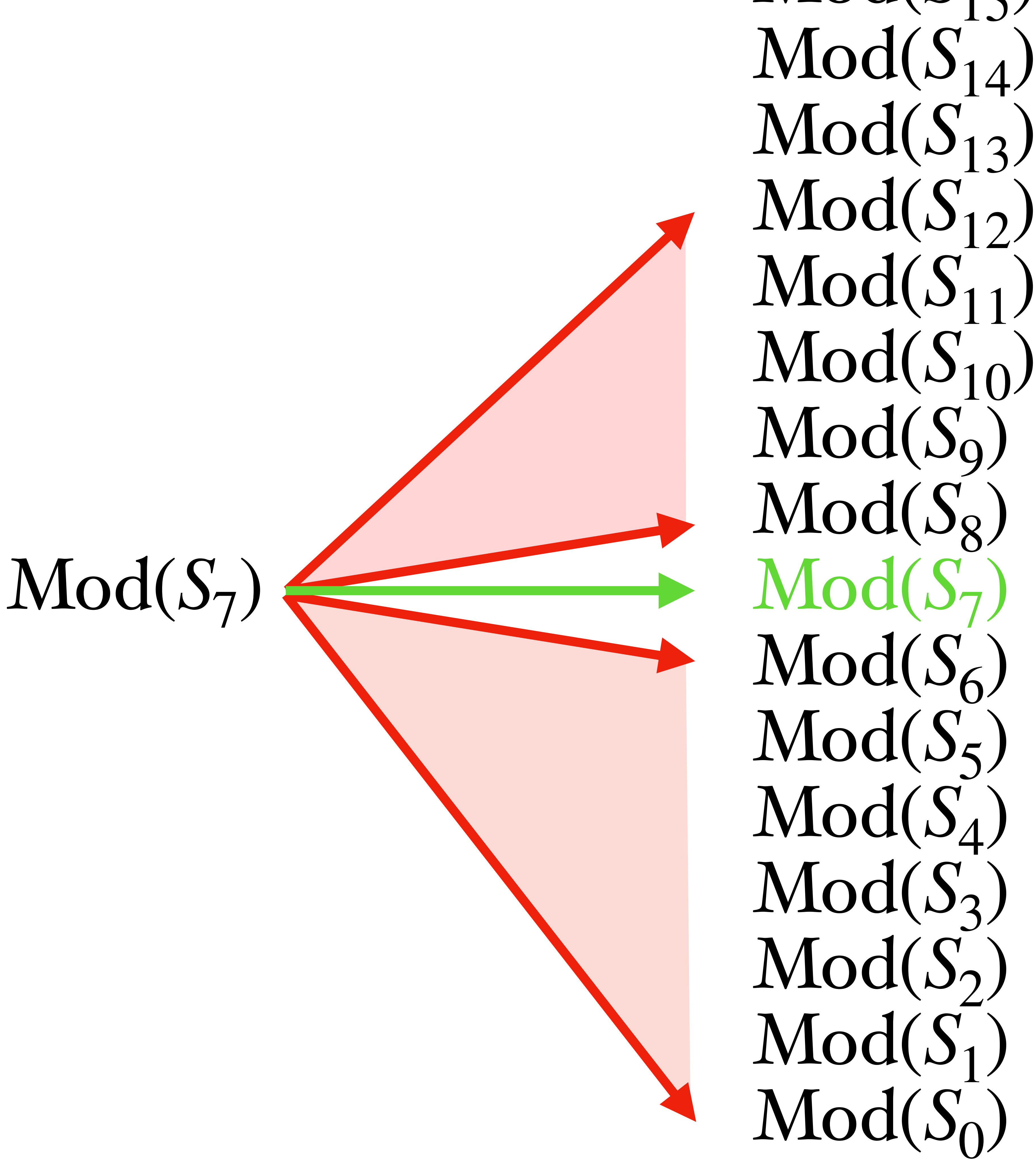

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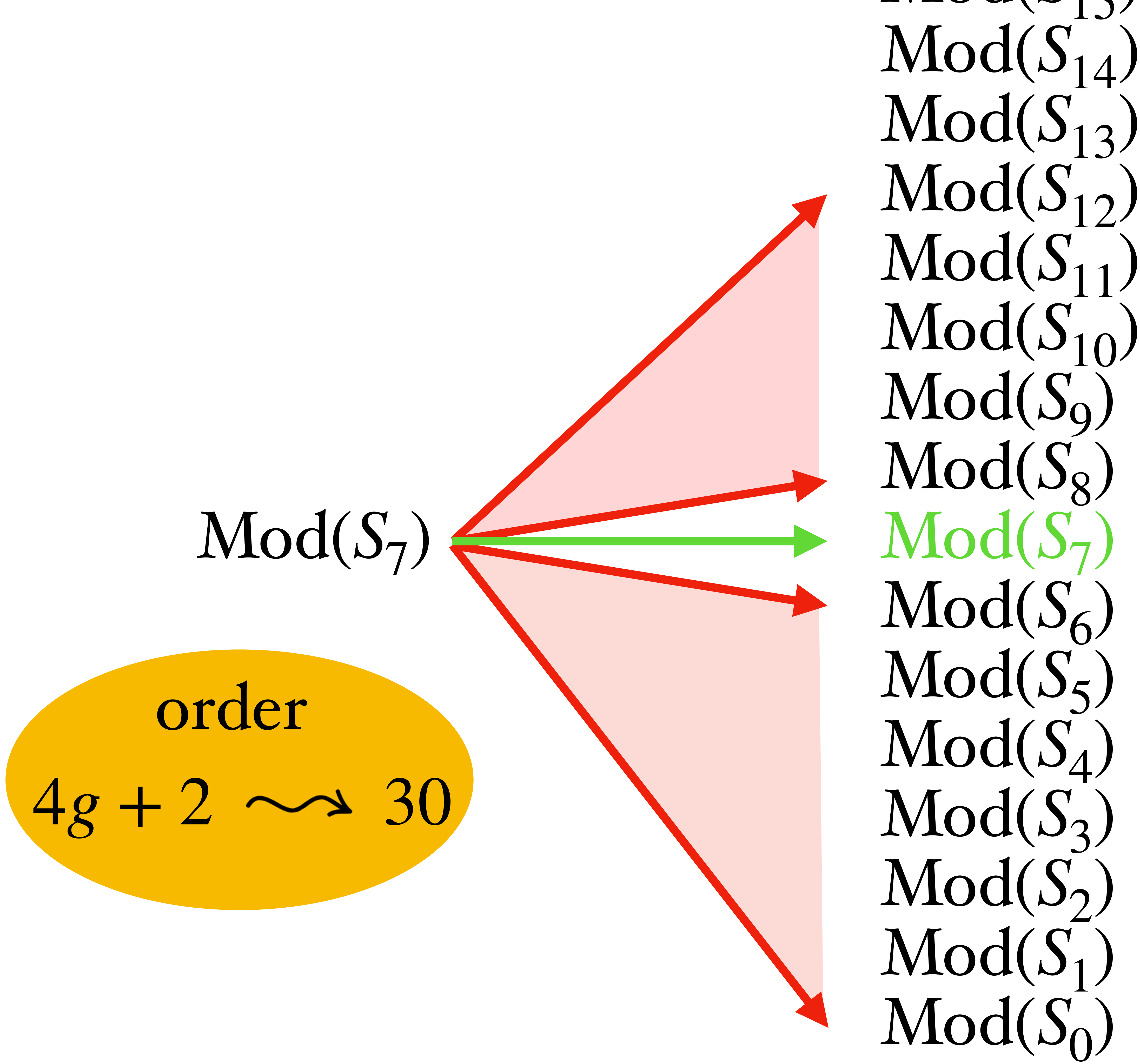
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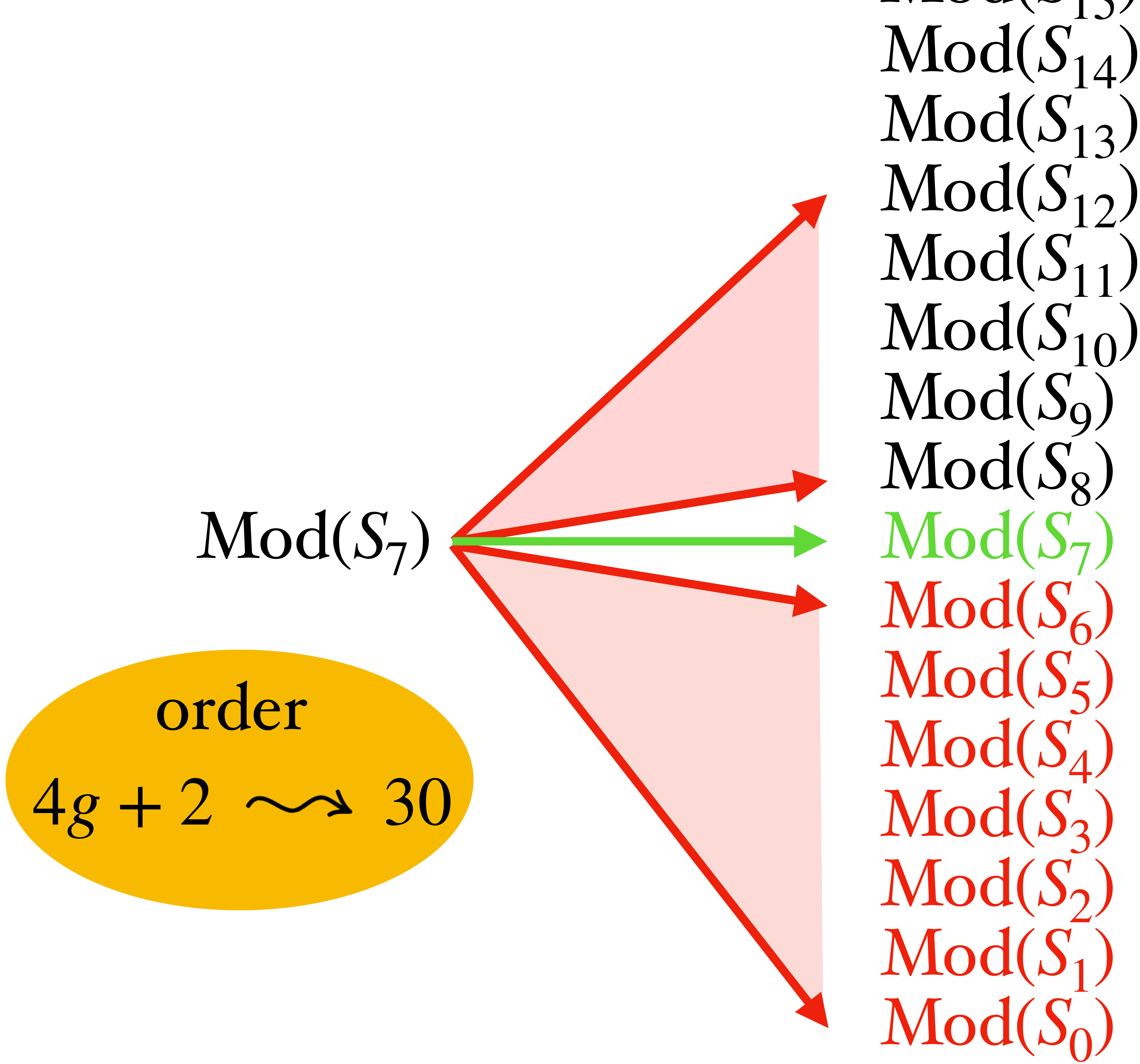
order
$4g+2 \rightsquigarrow 30$



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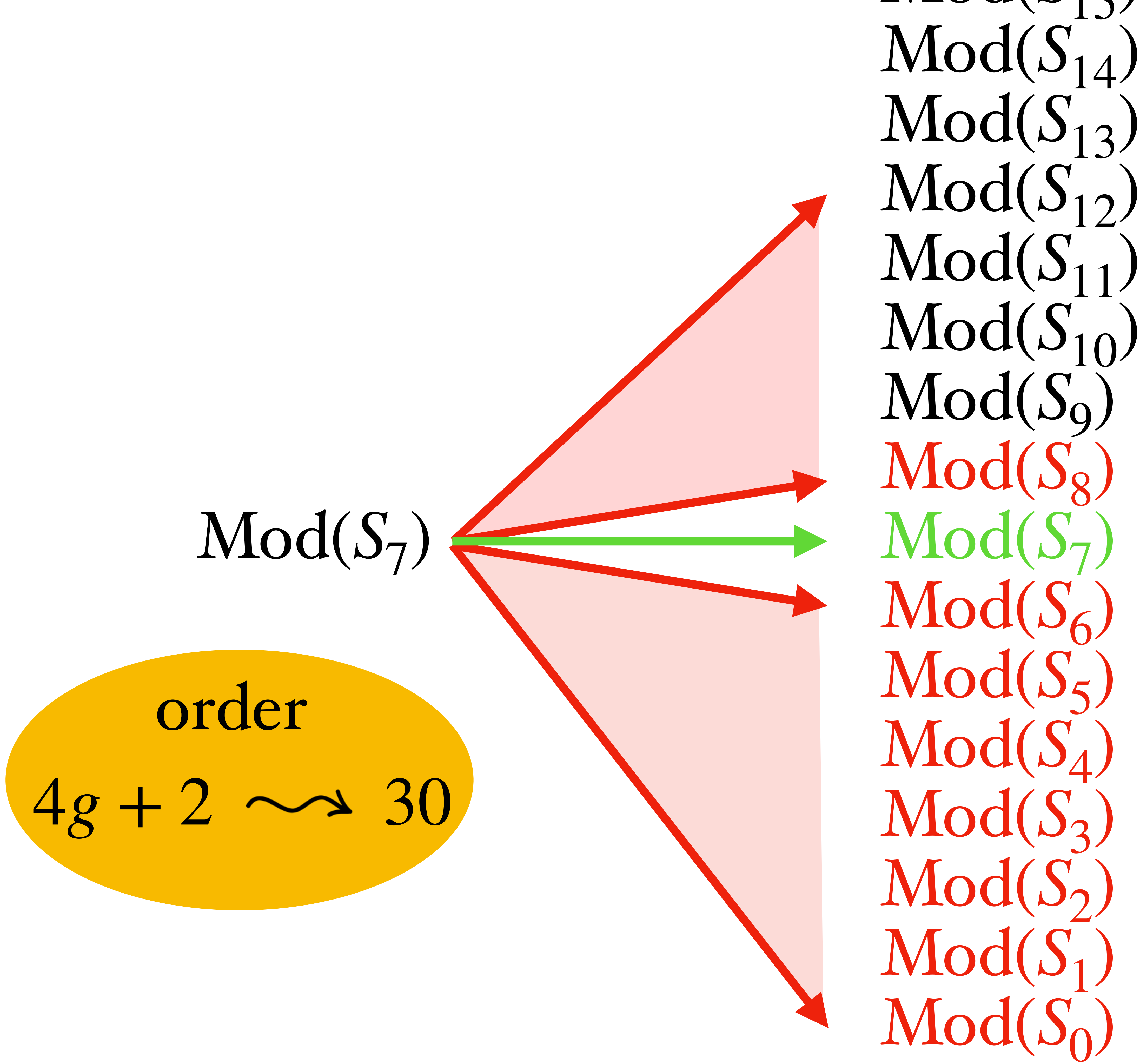

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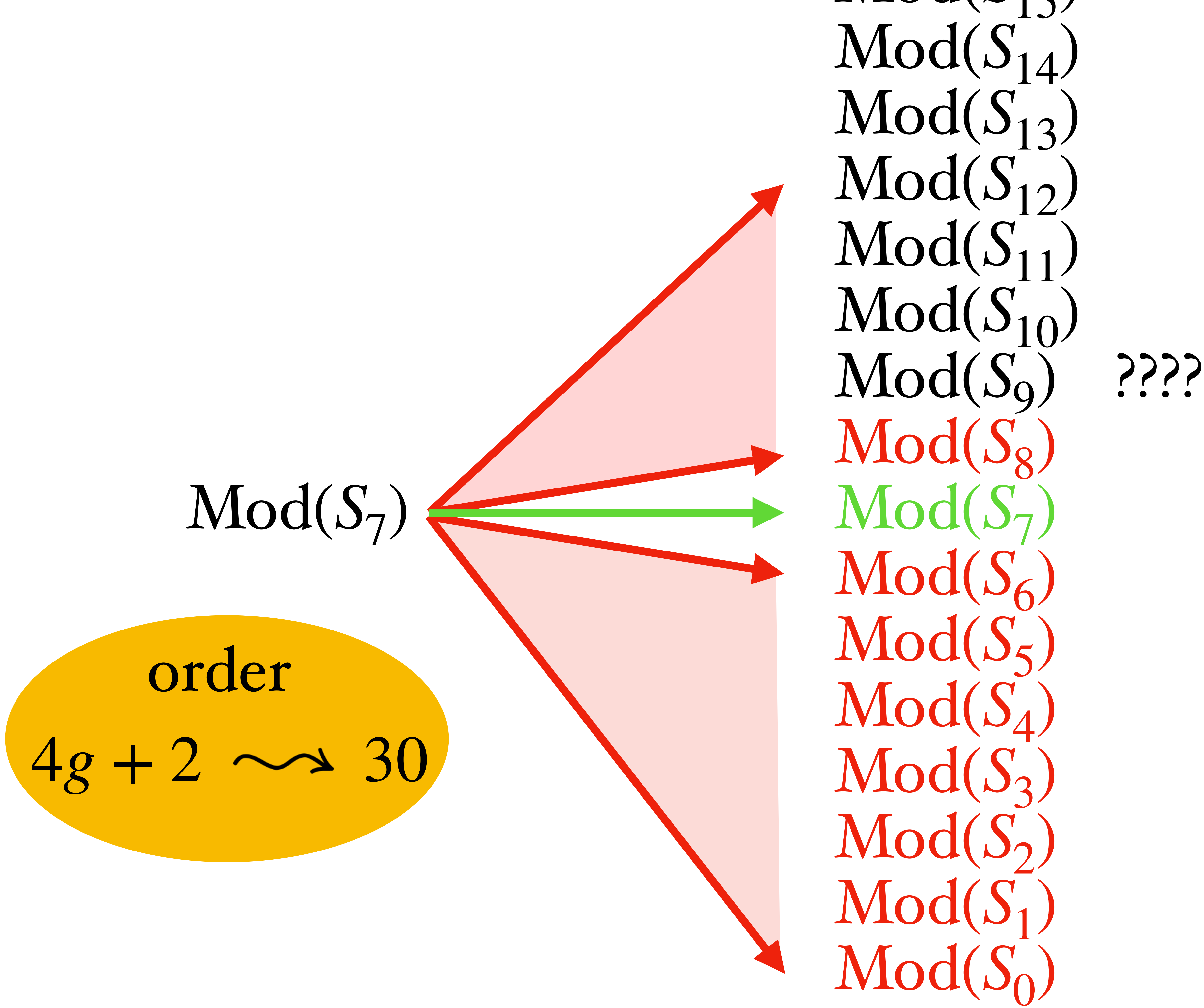

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Theorem (May–Zimmerman)

For $g \geq 3$ and odd, $\mathrm{Mod}(S_g)$ contains the first appearance of $C_4 \times D_g$

For $g \geq 2$ and even, $\mathrm{Mod}(S_g)$ contains the first appearance of DC_g

Theorem (May–Zimmerman)

For $g \geq 3$ and odd, $\mathrm{Mod}(S_g)$ contains the first appearance of $C_4 \times D_g$

For $g \geq 2$ and even, $\mathrm{Mod}(S_g)$ contains the first appearance of DC_g

Lemma (Chen–L)

These first appearances are the *only* appearances in the specified linear range.

[illegible]

Theorem 3 (Theorem 1.1, [18]). *For $g \geq 3$, every nontrivial periodic mapping class that is not a hyperelliptic involution normally generates* $\mathrm{Mod}(S_g)$.

It follows immediately from this theorem that any homomorphism from $\mathrm{Mod}(S_g)$ with a nontrivial nonhyperelliptic periodic element in its kernel is trivial. The strong constraints on homomorphisms provided by Theorem 3 are not, however, immediately applicable to the settings of Theorems 1 and 2. Under the hypotheses of each of these theorems, there are cases where, a priori, there could be a homomorphism to the target group where no nontrivial periodic mapping class of $\mathrm{Mod}(S_g)$ would lie in the kernel. For instance, for every periodic element in $\mathrm{Mod}(S_7)$, there is an element of $\mathrm{Mod}(S_9)$ of the same order. For Theorem 2, the situation is more extreme, as $\mathrm{Homeo}^+(\mathbb{S}^2)$ contains elements of every finite order. Despite these obstacles, we are still able to apply Theorem 3 to these situations by analyzing finite non-cyclic subgroups.

Homomorphisms between mapping class groups.
Aramayona–Souto proved a theorem constraining homomorphisms between the mapping class groups of surfaces $S_{g,n,b}$, which are surfaces of genus g with n punctures and b boundary components [2]. The mapping class groups they consider are pure, in that they require mapping classes to fix punctures and boundary components pointwise.

Theorem 4 (Aramayona–Souto, Theorem 1.1, [2]). *Let $S = S_{g,n,b}$ and $S' = S'_{g',n',b'}$, such that $g \geq 6$ and $g' \leq 2g-1$. If $g' = 2g-1$, suppose also that S' is not closed. Then every nontrivial homomorphisms $\phi : \mathrm{Mod}(S) \to \mathrm{Mod}(S')$ is induced by an embedding $S \to S'$.*

Aramayona–Souto also proved that the conclusion of this theorem holds when $g = g' \in \{4, 5\}$. They explain the necessity of their upper bound of $2g-1$ by observing that there is a "double embedding" homomorphism $\mathrm{Mod}(S_{g,0,1}) \to \mathrm{Mod}(S_{2g,0,0})$.

When restricted to the case of closed surfaces, Theorem 4 says that for $g \geq 6$ and $h < 2g-1$, every homomorphism $\phi : \mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$ is trivial, except for the possibility of an isomorphism in the case $g = h$. Some cases of this result for closed surfaces were previously known. The range $h < g$ was previously handled by Harvey–Korkmaz [14], while the case $h = g$ was shown by Ivanov–McCarthy under the further hypothesis that the homomorphism is injective [15].

Theorem 1 gives a new proof of Theorem 4 when restricted to the case where S and S' are closed surfaces and $g \neq g'$. Further, it is an easy consequence of the proof of Theorem 1 that we may replace S_h with a surface of genus h and arbitrarily many punctures and boundary components, since $\mathrm{Mod}(S_{h,n,b})$ has no additional finite subgroups compared to $\mathrm{Mod}(S_h)$. For this replacement, note that it is not necessary for us to restrict to $\mathrm{Mod}(S_{h,n,b})$ being pure. Our Theorem 1 also covers the additional small values of g of 3, 4, and 5, confirming an expectation that Aramayona–Souto state in their paper. We also extend their upper bounds for g' slightly when g is odd. This is possible because, for closed surfaces, their upper bound of $2g-1$ does not have the same natural justification that appears for surfaces with boundary.

It is important to note that, even for closed surfaces, some upper bound is necessary on h to ensure that the homomorphism is trivial. First, since $\mathrm{Mod}(S_g)$ is residually finite, it has a rich supply of finite quotients. As every finite group is a subgroup for some $\mathrm{Mod}(S_h)$, we obtain for all $g > 0$ non-trivial homomorphisms $\mathrm{Mod}(S_g) \to$

[illegible]

Lemma 7. *Let $g \geq 3$ and let $\psi : \mathrm{Sp}(S_g) \to G$ be a homomorphism with a nontrivial periodic element in its kernel. If G does not contain $(\mathbb{Z}/3)^g$ as a subgroup, then ψ is trivial.*

Proof. We proceed by constructing normal generators of $\mathrm{Sp}(2g, \mathbb{Z})$ and showing that at least one must be in the kernel of ψ.

The group $\mathrm{Sp}(2g, \mathbb{Z})$ contains $M = (C_3)^g$ as a subgroup, with generators m_1 to m_g of the form

$$m_i = \begin{bmatrix} I_{2i-2} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{2g-2i-4} \end{bmatrix}$$

where A is the 2×2 matrix

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

and I_n denotes the $n \times n$ identity matrix. Every nontrivial element in M is a normal generator of $\mathrm{Sp}(2g, \mathbb{Z})$, since it is the image of a normal generator of $\mathrm{Mod}(S_g)$, as we now show. The desired elements of $\mathrm{Mod}(S_g)$ are products of roots of Dehn twists about separating curves illustrated in Figure 1. For any nontrivial element $m \in M$, there is a corresponding mapping class $\tilde{m}$ and a nonseparating curve c so that c and $\tilde{m}(c)$ intersect exactly once. By the well-suited curve criterion, it follows that the mapping class $\tilde{m}(c)$ is a normal generator of $\mathrm{Mod}(S_g)$ [17, Lemma 2.2]; so too is its image in $\mathrm{Sp}(2g, \mathbb{Z})$, since normal generators descend to quotients.

As G contains no subgroup isomorphic to M, some normal generator of $\mathrm{Sp}(2g, \mathbb{Z})$ lies in the kernel of ψ; therefore ψ is trivial. □

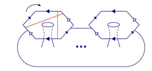

FIGURE 1. The surface S_g, where each of g hexagons with sides identified as indicated yields a handle. Each nontrivial element $\tilde{m} \in \tilde{M}$ consists of 1/3 or 2/3 rotations about some of the hexagons; we have $i(c, \tilde{m}(c)) = 1$ for some nonseparating simple closed curve c.

The following corollary can be thought of as an amplification of a theorem of Farb–Masur in the case of $\mathrm{Sp}(2g, \mathbb{Z})$; they show that for any irreducible lattice Γ in a semisimple lie group G of $\mathbb{R}$-rank at least two, the image of any homomorphism $\phi : \Gamma \to \mathrm{Mod}(S)$ is finite [12, Theorem 1.1].

Corollary 8. *Let $g \geq 3$ and let $h \leq 3^{g-1}$. Then every homomorphism $\psi : \mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Mod}(S_h)$ is trivial.*

Proof. This follows immediately from Lemma 10 by a result of Müller–Sarkar, who showed that the strong symmetric genus of $(C_3)^g$ is $1 + 3^{g-1} \cdot \mu_0(g)$, where $\mu_0(g) \geq 1$ when $g \geq 3$ [24, Section 9.1]. □

With these preliminaries established, the proof of Theorem 1 is straightforward.

Proof of Theorem 1. Let $g \geq 3$ be even and $g < h < 2g-1$. By Lemma 9, $\mathrm{Mod}(S_g)$ contains DC_g as a subgroup and $\mathrm{Mod}(S_h)$ does not. Then for any homomorphism $\phi : \mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$, a nontrivial periodic element f lies in the kernel. If f is not a hyperelliptic involution, we conclude that ϕ is trivial by Theorem 3. If f is a hyperelliptic involution, then ϕ factors through the symplectic representation. Since $h < 2g-1 \leq 3^{g-1}$, we conclude that ϕ is trivial by Corollary 11.

The proof for odd $g \geq 3$ proceeds in the same way using the finite group $C_4 \times D_g$. □

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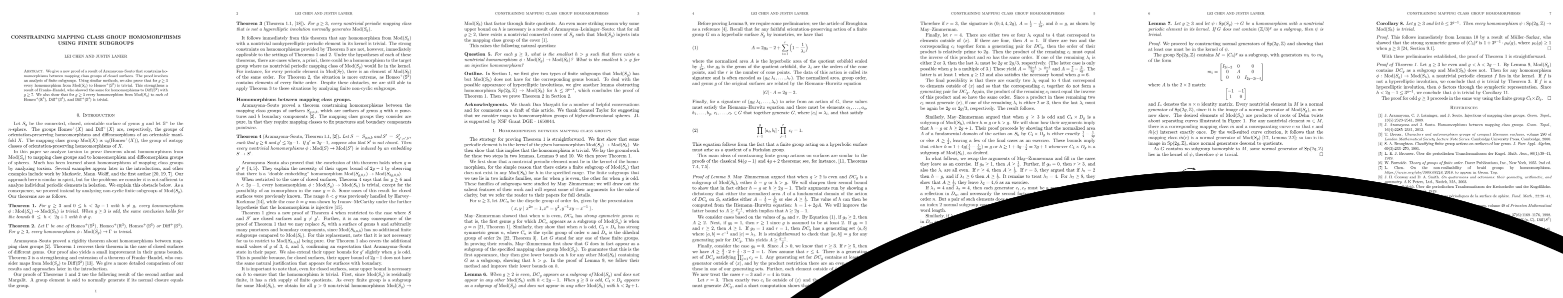
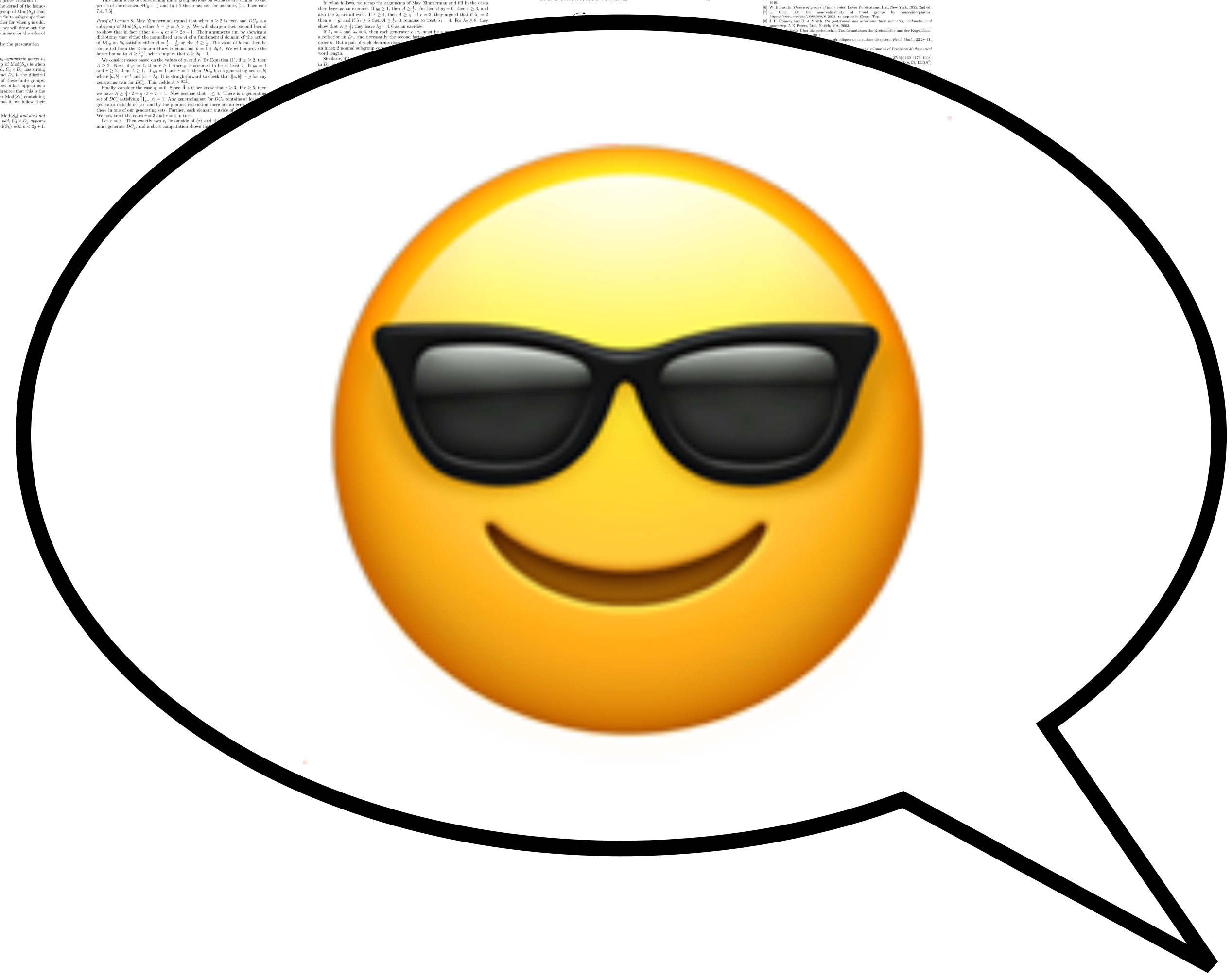
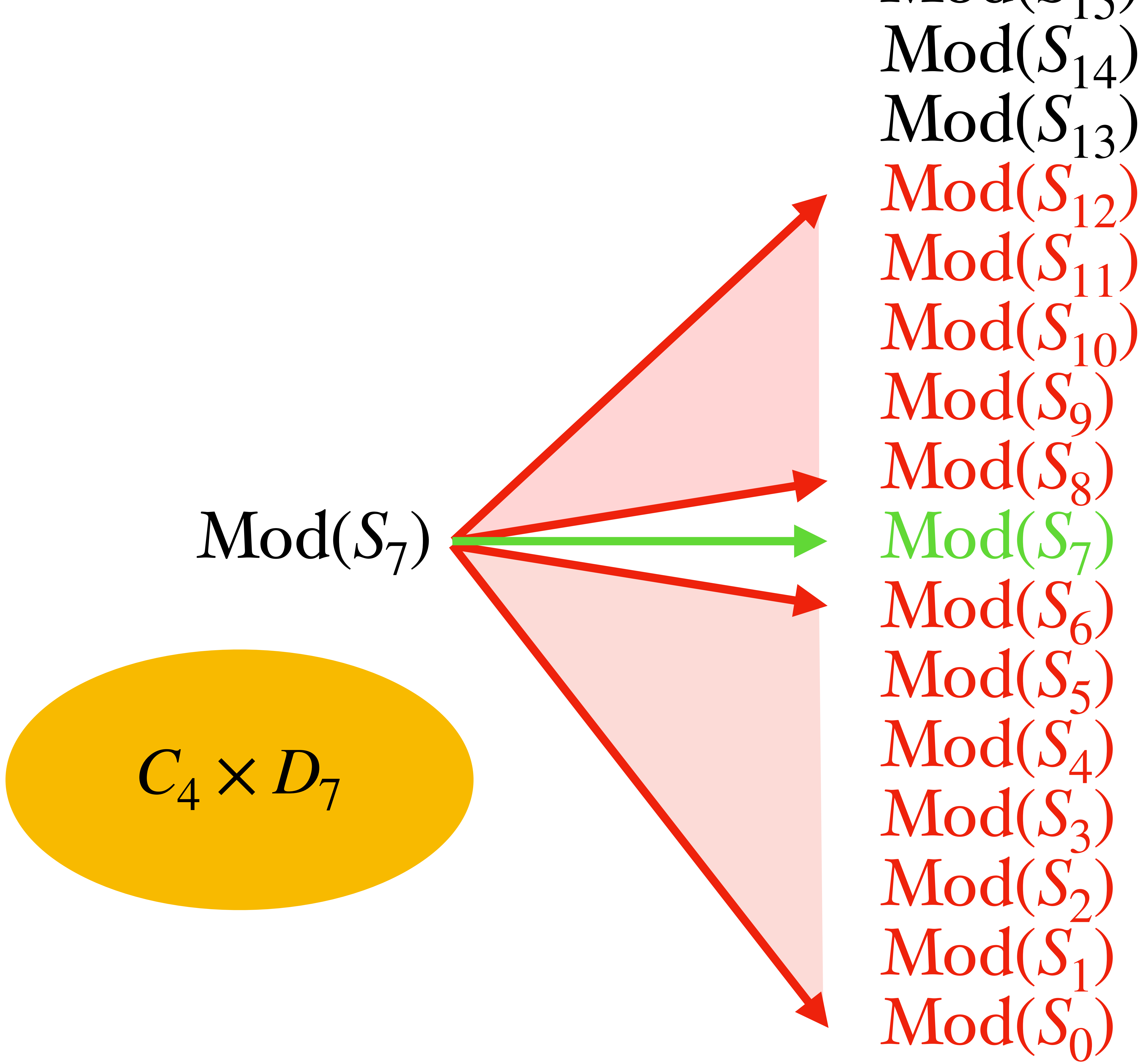

CONSTRAINING MAPPING CLASS GROUP HOMOMORPHISMS USING FINITE SUBGROUPS
LEI CHEN AND JUSTIN LANIER

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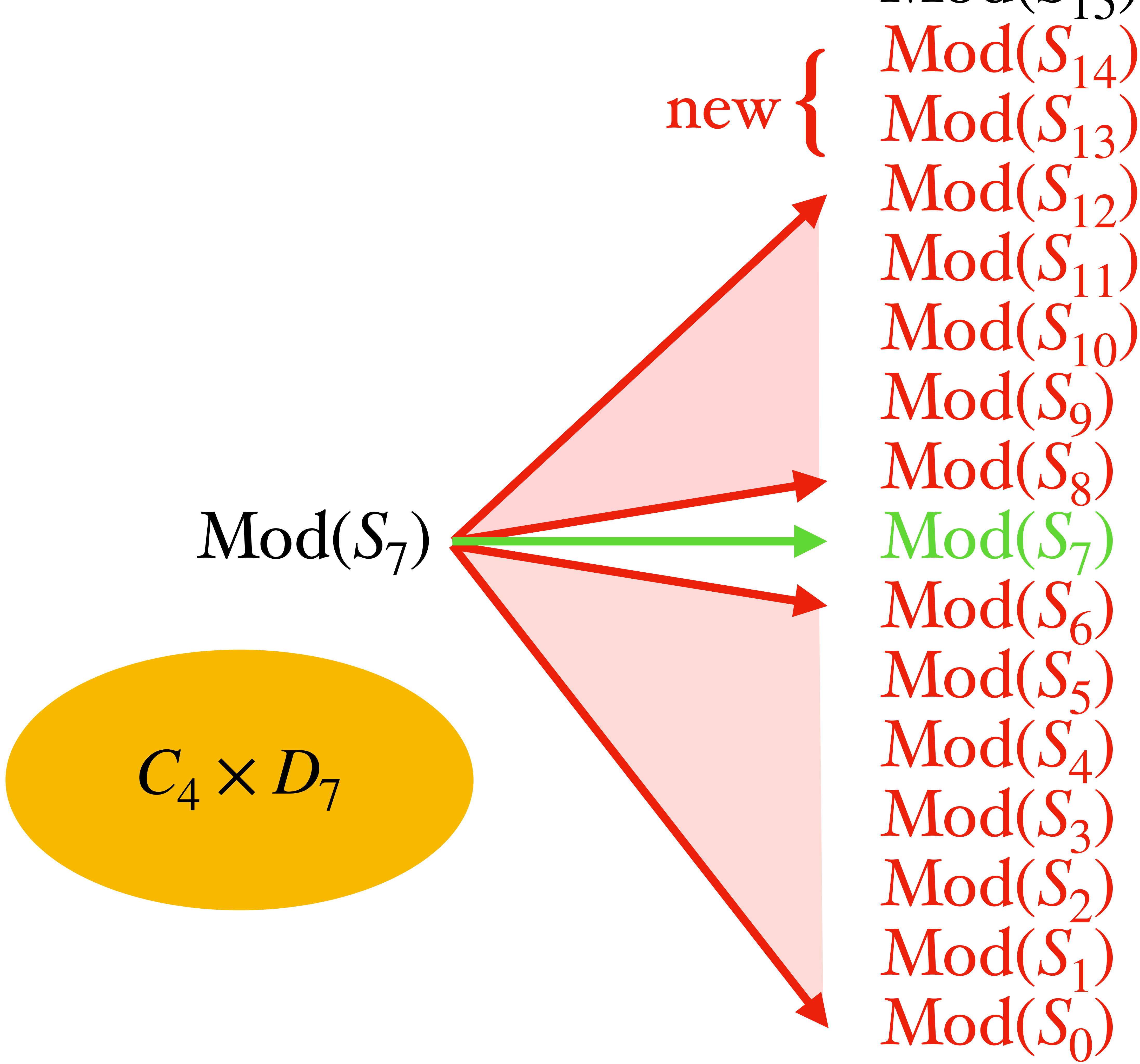
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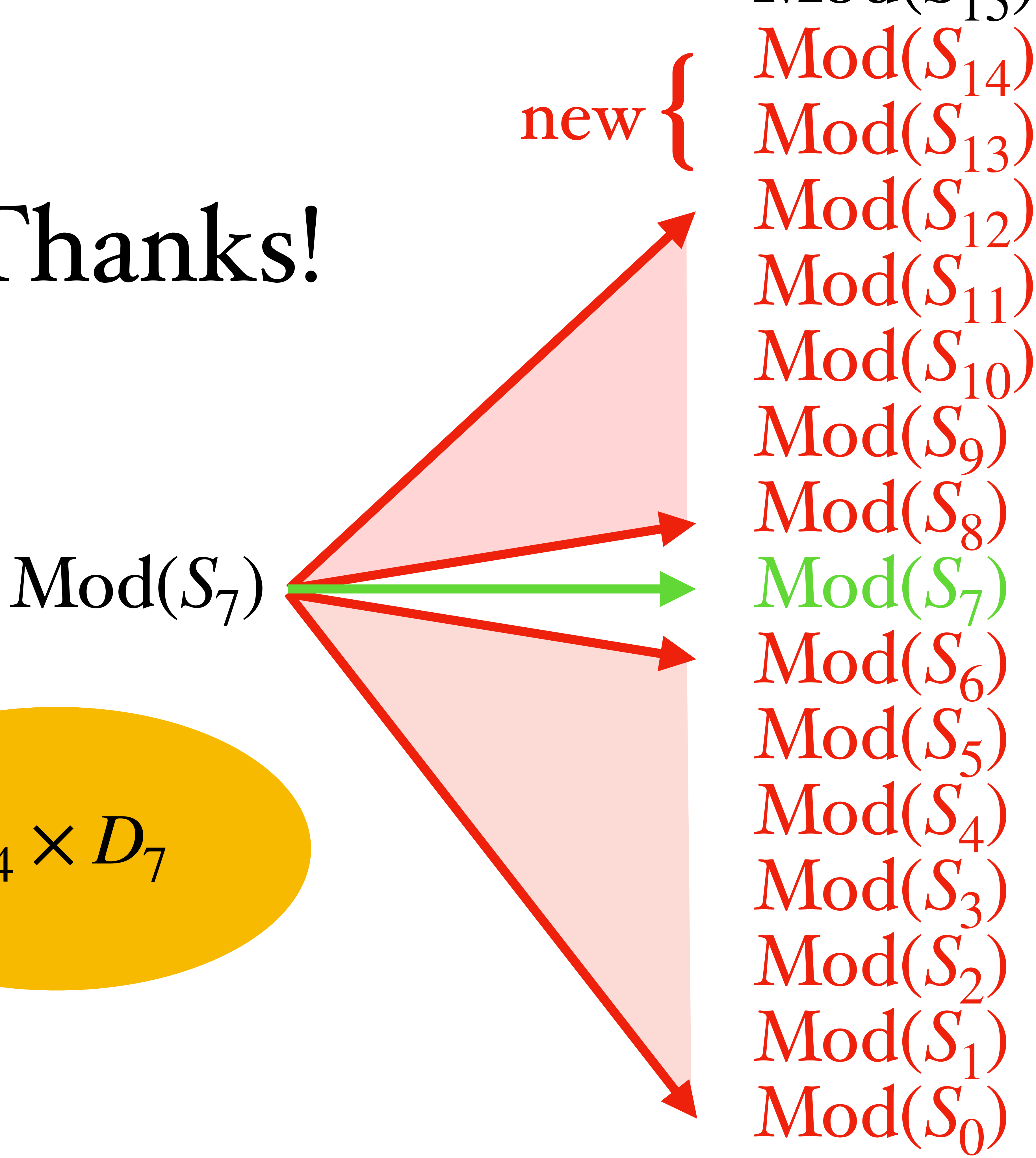

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Distribution of slope gaps for slit tori

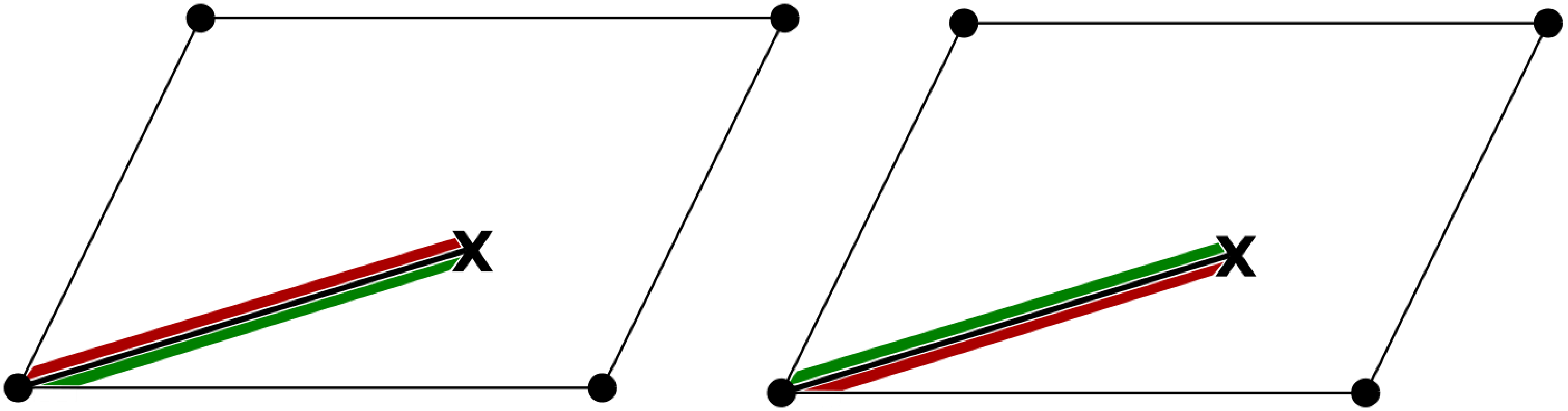
Anthony Sanchez

Tech Topology Conference

Georgia Institute of Technology

December 7th, 2019

Slit Tori



Genus 2 surface

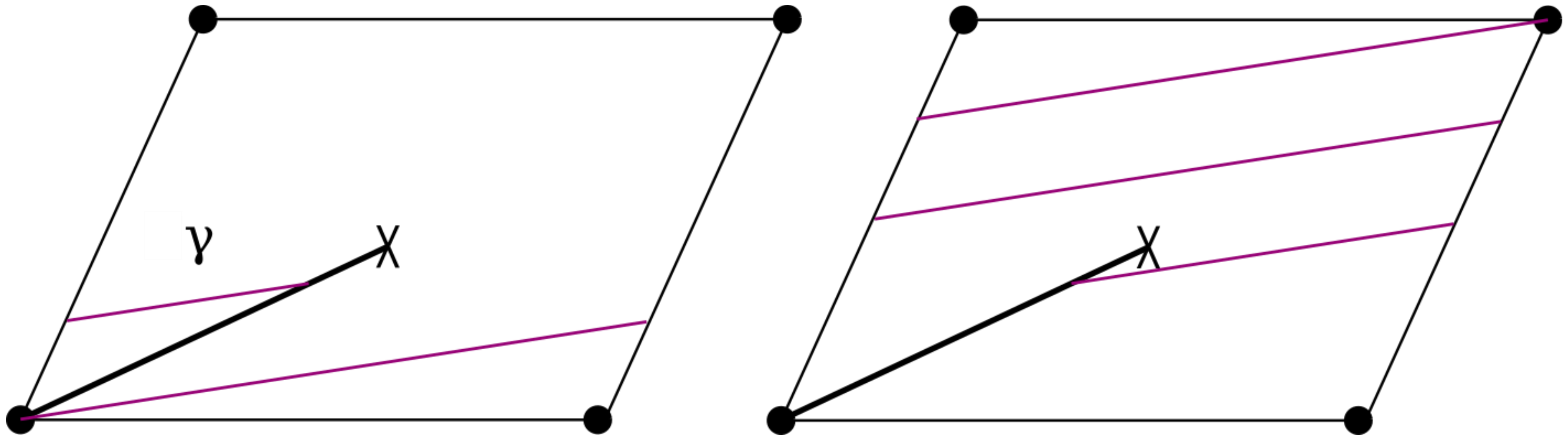
2 cone type singularities of angle 4π

Translation structure

Embedding into complex plane endows the surface with the holomorphic differential dz.

This allows us to measure lengths and gives a sense of direction.

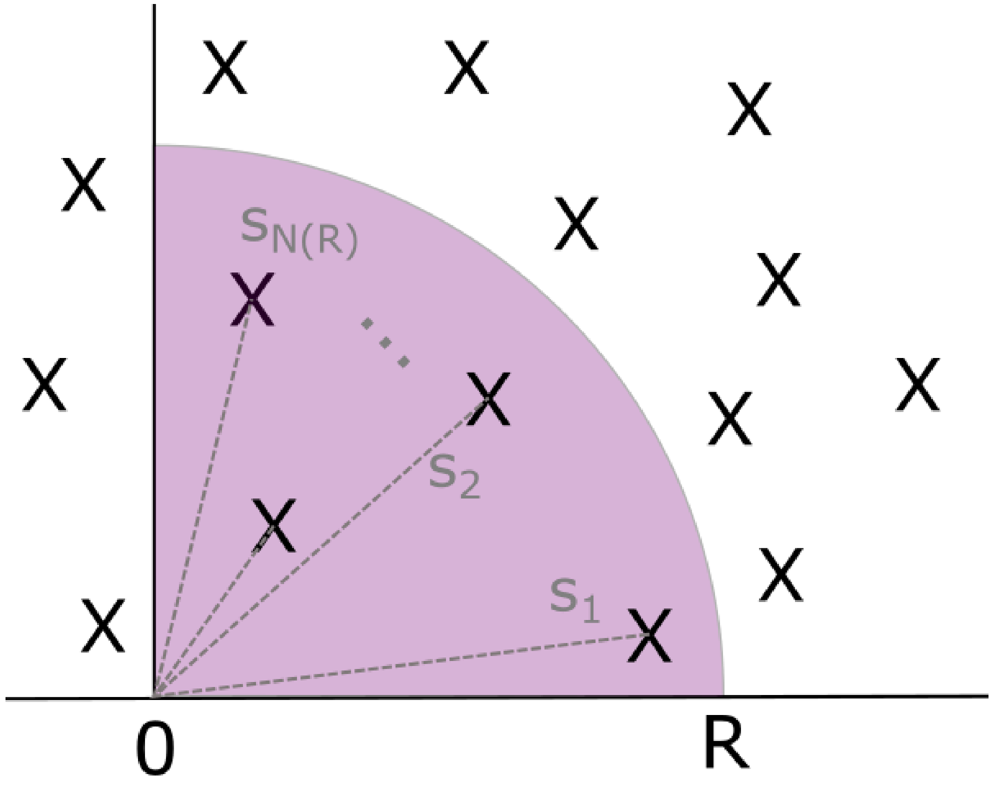
Holonomy vectors



Geodesics starting and ending at a cone type singularity are called *saddle connections.* The vector representing it is called the *holonomy vector*.

$$\boldsymbol{V_\gamma} := \int_{\boldsymbol{\gamma}} \boldsymbol{dz} = \begin{pmatrix} \mathbf{4} \\ \mathbf{1} \end{pmatrix}$$

How random are the holonomy vectors?



Random = gap distribution of slopes

Let Λ_ω denote the set of holonomy vectors

$$\boldsymbol{Slopes^R(\Lambda_\omega) = \{s_0 = 0 < s_1 < \cdots < s_{N(R)}\}}$$

$$\boldsymbol{\widetilde{Gaps}^R(\Lambda_\omega) = \{s_i - s_{i-1} \mid i = 1, \ldots, N(R)\}}$$

Gap distribution

Since $N(R) \sim \pi\, R^2$ it is natural to consider the normalized gaps

$$\boldsymbol{Gaps^R(\Lambda_\omega) = \left\{R^2(s_i - s_{i-1}) \mid i = 1, \dots, N(R)\right\}}$$

The *gap distribution* is given by the limit

$$\lim_{R\to\infty} \frac{|Gaps(\Lambda_\omega) \cap (c, d)|}{R^2}$$

What can we say about this limit?

Theorem (S. 2019)

There exists a density function f so that

$$\lim_{R\to\infty} \frac{|Gaps^R(\Lambda_\omega)\cap(c,d)|}{R^2} = \int_c^d f(x)\,dx$$

Moreover, f so that has a *quadratic tail* and *support at zero.*

Quadratic tail: There is a constant k so that

$$\lim_{t\to\infty} f(t)\cdot t^2 = k$$

Support at zero: For every positive ε we have

$$\int_0^{\varepsilon} f(x)\,dx > 0$$

Thank you!

Special thanks to:

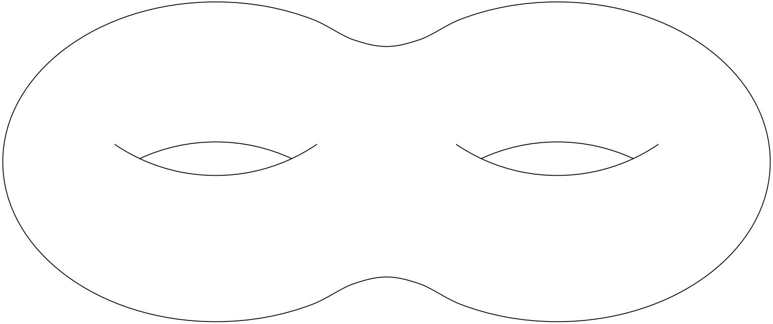
- Dr. Jayadev Athreya (My advisor)
- University of Washington
- Tech Topology Conference and Georgia Institute of Technology

Trees, dendrites, and the Cannon-Thurston map

Elizabeth Field
University of Illinois at Urbana-Champaign

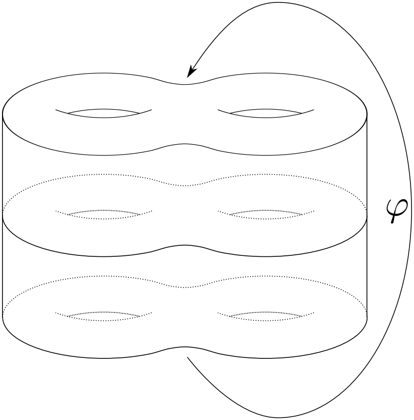
Tech Topology Conference
December 7, 2019

The original map of Cannon and Thurston



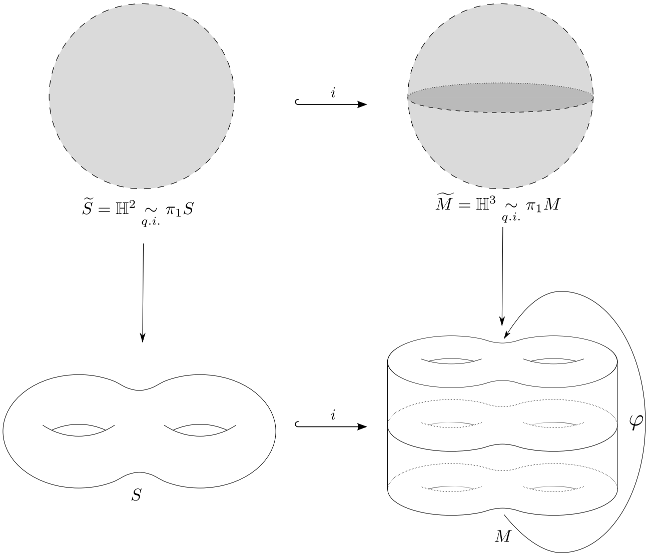
S - a genus $g \geq 2$, closed, oriented, hyperbolic surface

The original map of Cannon and Thurston



$$M = S \times [0,1] / \big((x,0) = (\varphi(x),1)\big)$$

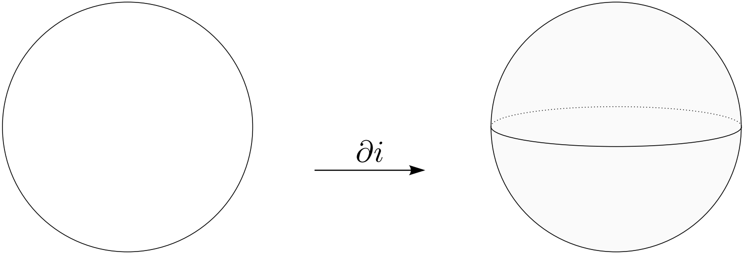
The original map of Cannon and Thurston



The original map of Cannon and Thurston

Theorem (Cannon-Thurston, 1984)

*The map $\partial\pi_1 S \xrightarrow{\partial i} \partial\pi_1 M$ is continuous and **surjective**.*



$\partial\pi_1 S = \partial\mathbb{H}^2 = \mathbb{S}^1$

$\partial\pi_1 M = \partial\mathbb{H}^3 = \mathbb{S}^2$

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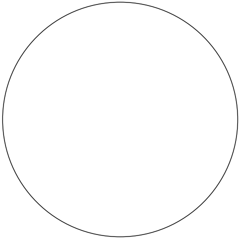
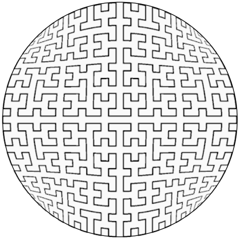


$\partial\pi_1 S = \partial\mathbb{H}^2 = \mathbb{S}^1$

$\xrightarrow{\partial i}$



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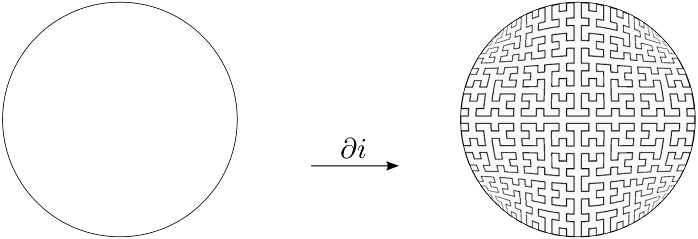


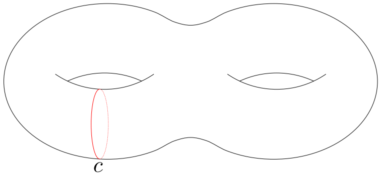
$\partial\pi_1 S = \partial\mathbb{H}^2 = \mathbb{S}^1$

$\partial\pi_1 M = \partial\mathbb{H}^3 = \mathbb{S}^2$

Definition

Let H and G be hyperbolic groups with $H \leq G$. If the inclusion map $i : H \to G$ extends to a continuous map $\partial i : \partial H \to \partial G$, this map is called the *Cannon-Thurston map*.

Topology of the original Cannon-Thurston map



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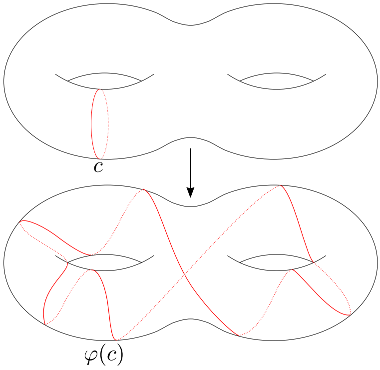

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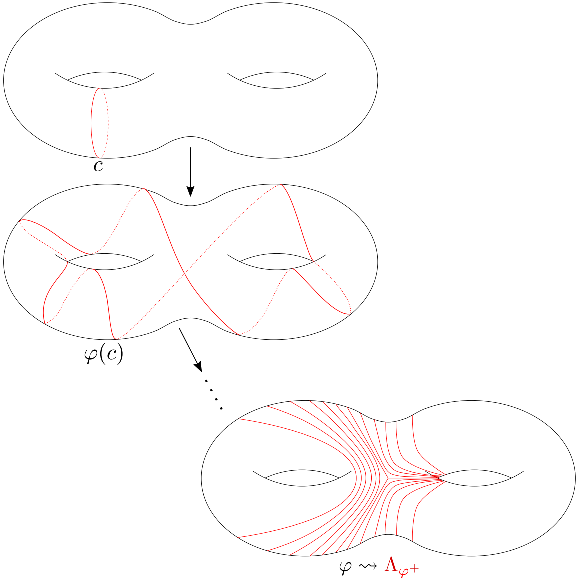
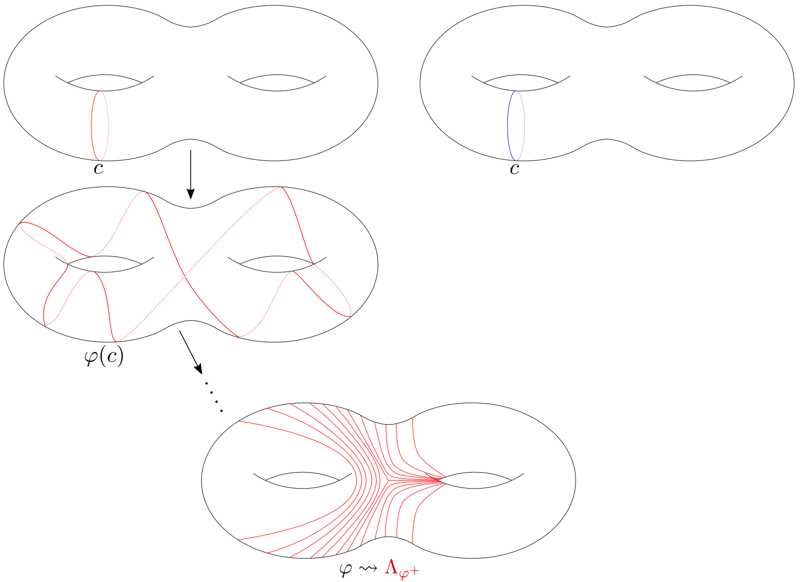


Geodesic ending lamination

Topology of the original Cannon-Thurston map



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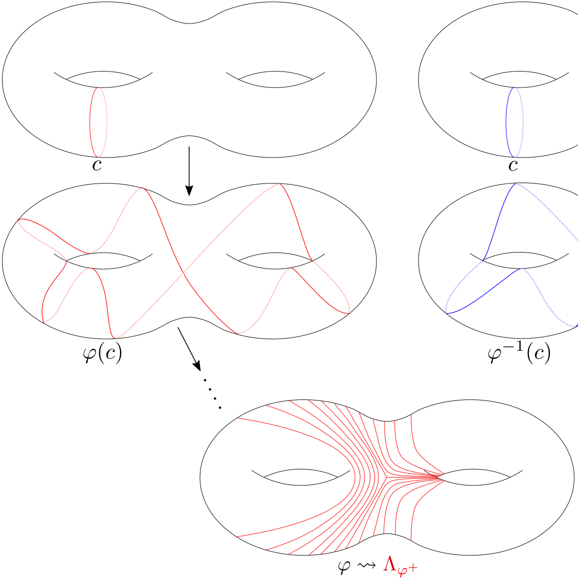

Geodesic ending lamination

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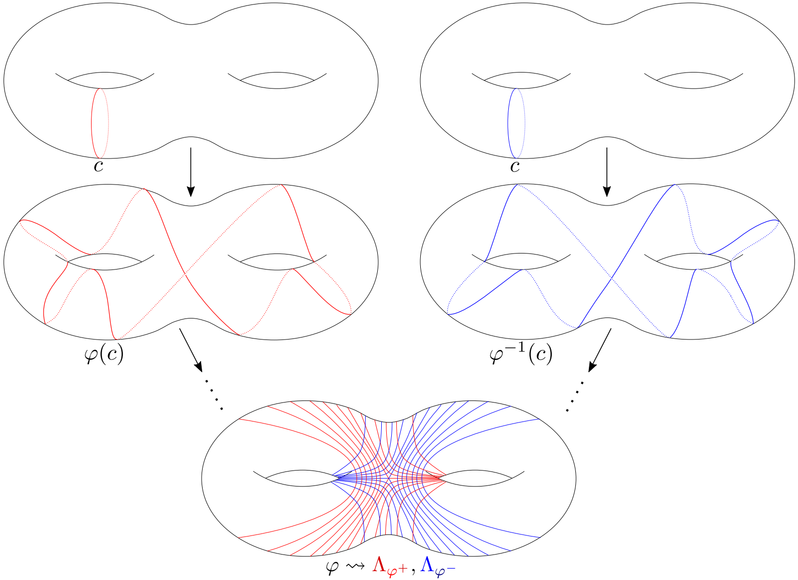
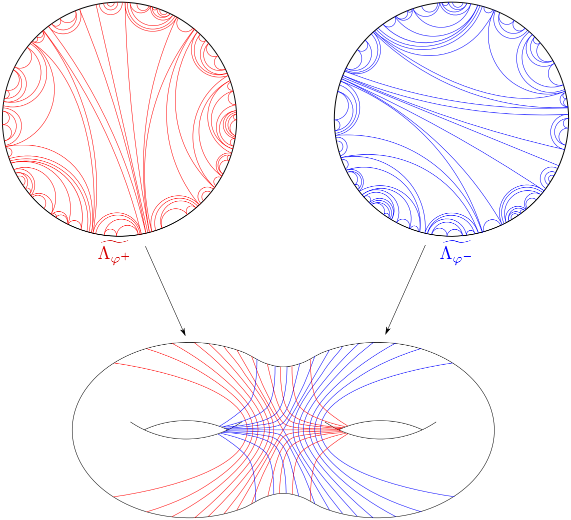


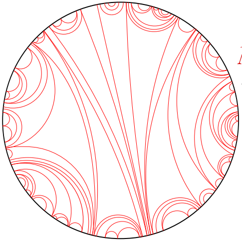
Geodesic ending laminations

Topology of the original Cannon-Thurston map



$\varphi \rightsquigarrow \Lambda_{\varphi^+}, \Lambda_{\varphi^-}$

Topology of the original Cannon-Thurston map



$\widetilde{\Lambda_{\varphi^+}}$

$\partial\pi_1 S \cup \pi_1 S = \mathbb{S}^1 \cup \mathbb{H}^2$

$\partial\pi_1 M = \partial\mathbb{H}^3 = \mathbb{S}^2$

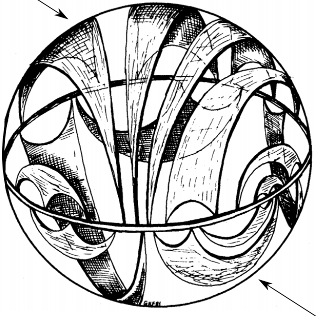
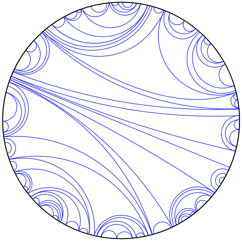


Illustration by George Francis

$\partial\pi_1 S \cup \pi_1 S = \mathbb{S}^1 \cup \mathbb{H}^2$



$\widetilde{\Lambda_{\varphi^-}}$

Topology of the original Cannon-Thurston map

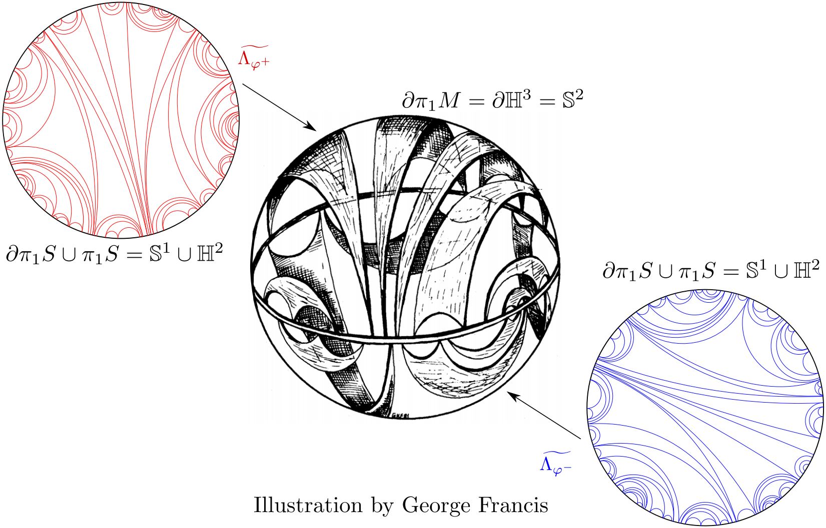
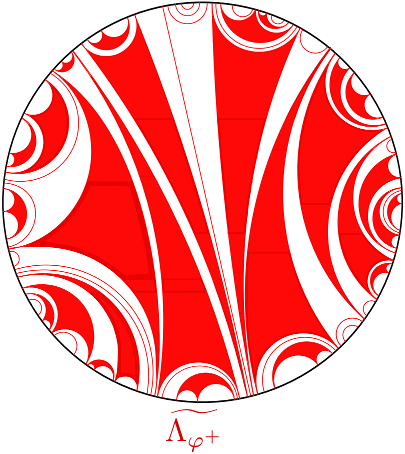


Illustration by George Francis

$$\partial i(\partial \pi_1(S)) = \mathbb{S}^2 / (\widetilde{\Lambda_{\varphi^+}} \cup \widetilde{\Lambda_{\varphi^-}})$$

What is $\partial\pi_1(S)/\widetilde{\Lambda_{\varphi^+}}$?



$\rightsquigarrow$???

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$\rightsquigarrow$ dual $\mathbb{R}$-tree T_{φ^+}

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$\widetilde{\Lambda_{\varphi^+}}$

$\rightsquigarrow$ dual $\mathbb{R}$-tree T_{φ^+}

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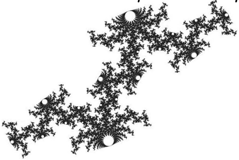

Definition

A *dendrite* is a compact, connected, locally connected metric space with no simple closed curves.

How can we generalize this?

Cannon and Thurston's example:

$$1 \to \pi_1 S \to \pi_1 M \to \langle \varphi \rangle \to 1$$

The Cannon-Thurston map $\partial i : \partial \pi_1 S \to \partial \pi_1 M$ exists and is surjective.

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General case [Mitra, 1998]: Let H, G, and Q be infinite, hyperbolic groups with

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Theorem (F.)

$\partial H / \Lambda_z$ is a dendrite (a compact, tree-like topological space).

An algorithm for an upper bound on splitting genus

Christopher Anderson

University of Miami

canders@math.miami.edu

Dec 7th 2019

Notation and Definitions

- $L = L_1 \cup L_2 \subset S^3$ a 2-component link
- $X = S^3 \backslash \mathcal{N}(L)$
- $\rho : \widetilde{X} \to X$ the universal abelian covering map
- Group of deck transformations $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^2 =< s, t >$
- $\Lambda \cong \mathbb{Z}H_1(X) \cong \mathbb{Z}[s, s^{-1}, t, t^{-1}]$

The multivariable Alexander polynomial and $H_2(\widetilde{X}, \mathbb{Z})$

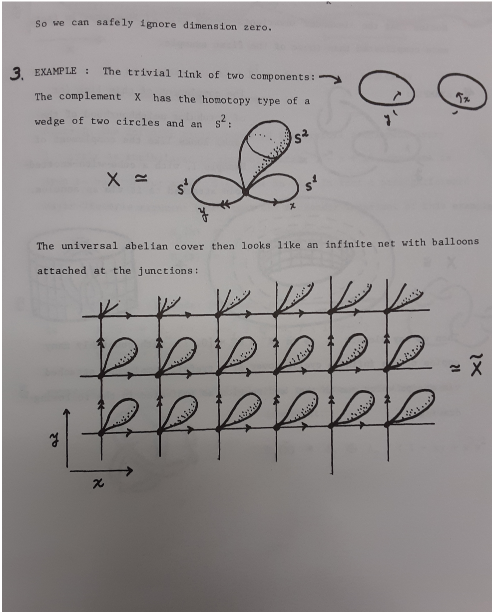
- Thm (e.g. Cochrane, 70): $\Delta_L(s,t) = 0 \Leftrightarrow H_2(\widetilde{X}, \mathbb{Z}) \cong \Lambda$ when regarded as a Λ-module.
- Definition:

$$g_{split} = min\{g(S) : \text{S surface and [S] generates } H_2(\widetilde{X}, \mathbb{Z})\}$$

- Thm: $g_{split} = 0$ if and only if L is a split link.
- Thm (A, Baker, in progress) $g_{split} = 1$ if and only if L is non-split and X contains an embedded essential torus that separates a pair of disjoint Seifert surfaces for L_1 and L_2

Ex: the 2-component unlink

Credit: *Knots and Links* by Dale Rolfsen



So we can safely ignore dimension zero.

3. EXAMPLE : The trivial link of two components:

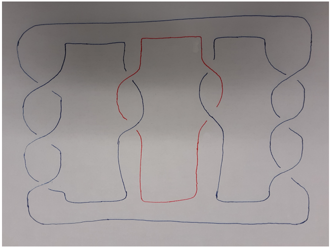
The complement X has the homotopy type of a wedge of two circles and an S^2:

$X \simeq$

The universal abelian cover then looks like an infinite net with balloons attached at the junctions:

$\simeq \tilde{X}$

Ex: a pretzel link



- Pretzel Link $P(3, -2, 2, -3)$
- Since it is hyperbolic, $g_{split} \geq 2$
- We show by construction that $g_{split} \leq 2$.
- So $g_{split} = 2$.

Establishing an upper bound

- Goal: Construct surface $\Sigma \subset \widetilde{X}$ s.t. $[\Sigma]$ generates $H_2(\widetilde{X}, \mathbb{Z})$
- May not be of minimal genus, so only gives upper bound
- Assume L is non-split so X is a $K(\pi_1(X), 1)$-space

Getting a well-behaved 2-complex

Want to find 1-vertex 2-complex C s.t.

- C is constructed from a presentation of $\pi_1(X)$
- $C \hookrightarrow X$ and X def. retracts to C.
- The homology class of every 1-cell is s or t

Some examples

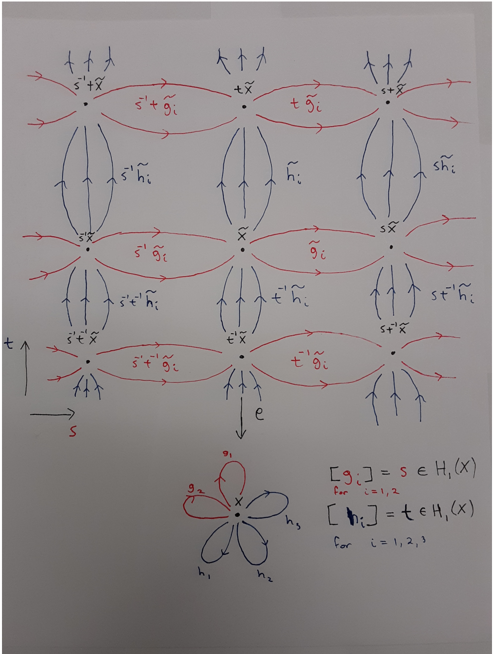
- Wirtinger presentation
- Bridge presentations

Lifting C

C lifts nicely to $\rho^{-1}(C) \cong \widetilde{C}$

- 0-cells lift to $\mathbb{Z}^2$ lattice
- 1-cells lift to horizontal or vertical edges connecting lattice points
- Abelianized Fox derivatives (plus more) tell us how to attach 2-cells

Illustrations



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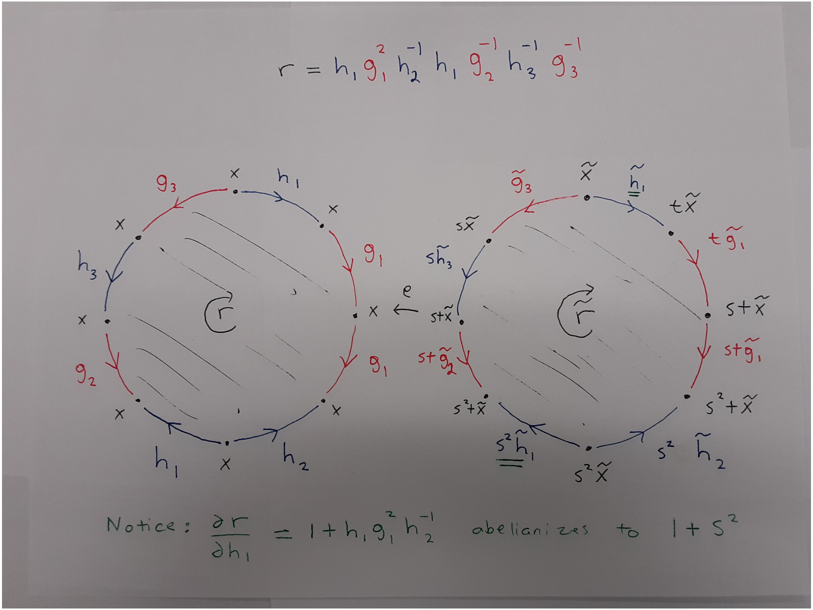


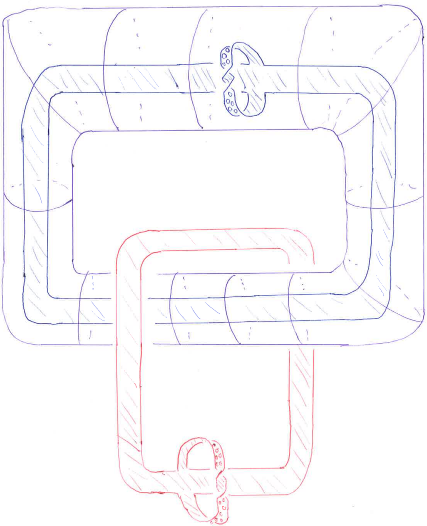
Finding a generator

- Alexander matrix describes the boundary map $\partial : \widetilde{C}_2 \to \widetilde{C_1}$
- $Ker(\partial) = H_2(\widetilde{C}, \mathbb{Z}) \cong H_2(\widetilde{X}, \mathbb{Z})$.
- Can find a generator of kernel by reducing it to reduced row echelon form

Finding a surface

- Generator is 2-cycle Σ
- Σ is not a surface...
- ...but we may find a surface $\Sigma' \subset \mathcal{N}(\widetilde{C}) \subset \widetilde{X}$ that carries the same homology class
- There is some choice involved in correcting S to a surface
- Can identify it by Euler Characteristic

Thank You!



Annular Rasmussen invariants: Properties and 3-braid classification

Gage Martin

Boston College

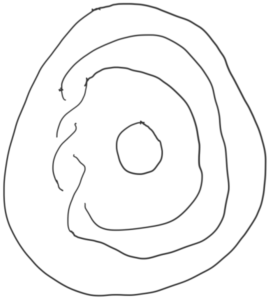
December 7th, 2019

Land Acknowledgement

As part of reflecting on the continuing legacy of colonialism and genocide here in the United States we should acknowledge that we are meeting on the stolen territory of the Muscogee people.

Annular Links

Annular Links



An Annular Link

Filtrations on Khovanov-Lee homology

- Khovanov-Lee homology carries a $\mathbb{Z}$ filtration used by J. Rasmussen to define the s invariant.

Filtrations on Khovanov-Lee homology

- Khovanov-Lee homology carries a $\mathbb{Z}$ filtration used by J. Rasmussen to define the s invariant.
- Working with an annular link adds an additional $\mathbb{Z}$ filtration on Khovanov-Lee homology.

What can you do with a $\mathbb{Z} \oplus \mathbb{Z}$ filtered complex?

- From knot Floer homology there is $\Upsilon_K(t)$ by Ozsváth-Stipsicz-Szabó and Livingston.

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- From annular-Khovanov-Lee homology there is $d_t(L)$ by Grigsby-A. Licata-Wehrli

What can you do with a $\mathbb{Z} \oplus \mathbb{Z}$ filtered complex?

- From knot Floer homology there is $\Upsilon_K(t)$ by Ozsváth-Stipsicz-Szabó and Livingston.
- From annular-Khovanov-Lee homology there is $d_t(L)$ by Grigsby-A. Licata-Wehrli
- Variants of this construction have been used by many people to define invariants of links, including Chakraborty, Lewark-Lobb, Sarkar-Seed-Szabó, and Truong-Zhang.

Why should we care about the d_t invariant?

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- $d_t(L)$ is an annular concordance invariant
- $d_t(\widehat{\beta})$ can detect right-veering, non-quasipositive braids
- There are connections between $d_t(\widehat{\beta})$ and transverse invariants of $\widehat{\beta}$ defined from Khovanov homology.

Restrictions on $d_t(\widehat{\beta})$ and $\Upsilon_K(t)$

Theorem (M.)

For a fixed braid index n, there are only finitely many possibilities for $d_t(\widehat{\beta})$ and a method for listing them all, where β is any n-braid.

Restrictions on $d_t(\widehat{\beta})$ and $\Upsilon_K(t)$

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Theorem (M.)

For a fixed concordance genus m, there are only finitely many possibilities for $\Upsilon_K(t)$ *and a method for listing them all, where K is any knot of concordance genus m.*

Computation of d_t and s invariants of 3-braids

Theorem (M.)

For any 3-braid β, it is possible to explicitly read off $d_t(\widehat{\beta})$ and $s(\widehat{\beta})$ from a distinguished representative of the conjugacy class of β.

Proof Sketch

- Express 3-braids in their Murasugi conjugacy form

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- Find enough 3-braids where it is “easy” to compute the d_t invariant

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- Express 3-braids in their Murasugi conjugacy form
- Find enough 3-braids where it is "easy" to compute the d_t invariant
- Use cobordisms to compute the d_t invariants of all other 3-braids

Thank You

Recognizing Pseudo-Anosov Braids in $\mathrm{Out}(W_n)$

Rylee Lyman, Tufts University

Tech Topology IX, Dec 7 2019

What is $\mathrm{Out}(W_n)$?

The free Coxeter group of rank n:

$$W_n = (\mathbb{Z}/2\mathbb{Z})^{*n} = \langle a_1, \ldots, a_n \mid a_i^2 = 1 \rangle.$$

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"Nielsen-like" generators:

$$\tau_{ij} \begin{cases} a_i \mapsto a_j \\ a_j \mapsto a_i \\ a_k \mapsto a_k & k \neq i, j \end{cases} \qquad \chi_{ij} \begin{cases} a_j \mapsto a_i a_j a_i \\ a_k \mapsto a_k & k \neq j. \end{cases}$$

A Classification Theorem

Theorem (L, '19)

*Every outer automorphism $\varphi \in \mathrm{Out}(W_n)$ may be represented by a homotopy equivalence $f\colon G \to G$ of a W_n-**orbigraph** with special properties called a **relative train track map.***
*If φ is (fully) **irreducible,** the special homotopy equivalence is nicer and is called a **train track map.***

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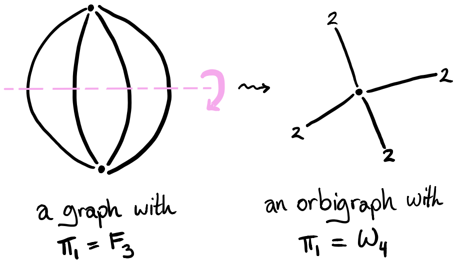
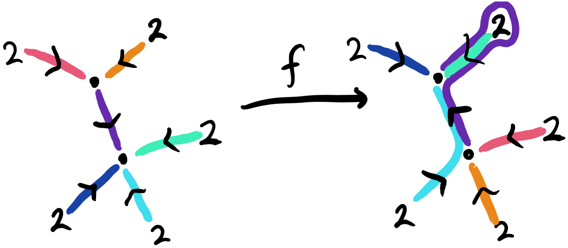
Builds on work of Bestvina, Feighn and Handel for $\mathrm{Out}(F_n)$.

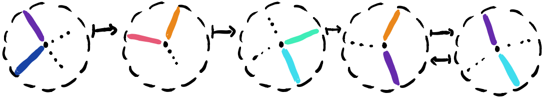


A Train Track Map

A homotopy equivalence $f\colon G \to G$ is a **train track map** when for each edge $e \in G$, the kth iterate $f^k|_e$ is an immersion for all $k \geq 1$.



This is a train track map!



The Project

Pseudo-Anosov mapping class is to **Pseudo-Anosov homeomorphism** as **fully irreducible outer automorphism** is to **train track map.**

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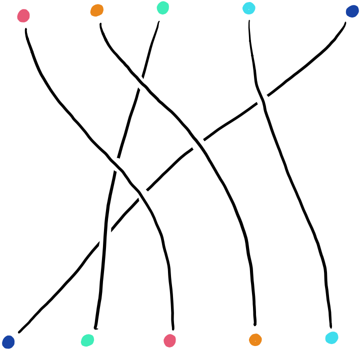
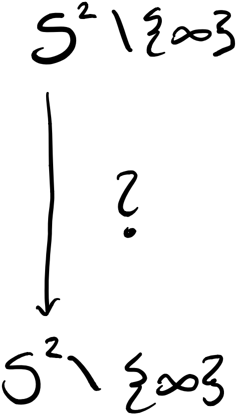
Theorem (L, In Progress)

*If $\varphi \in \mathrm{Out}(W_n)$ is fully irreducible, it is either **hyperbolic** or φ^k can be represented as a pseudo-Anosov braid on an orbifold with one boundary component with orbifold fundamental group W_n for some $k \geq 1$.*

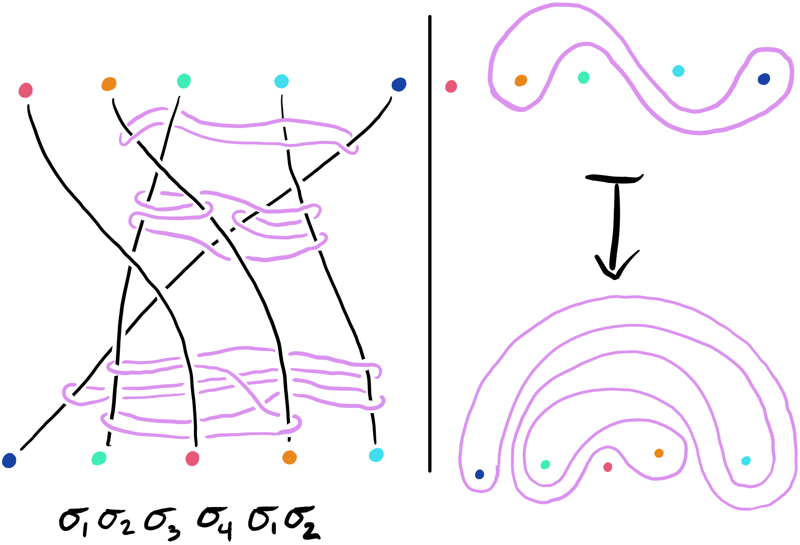
Braids As Mapping Classes



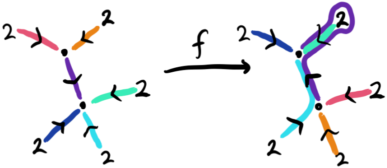
$\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_1 \sigma_2$



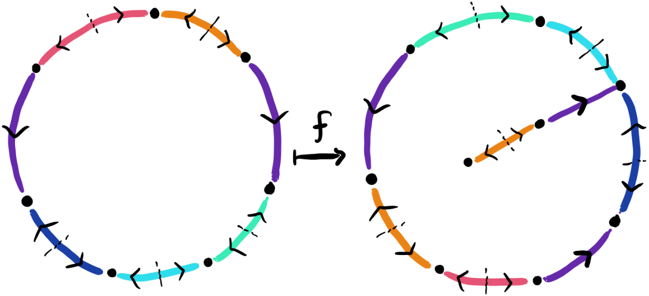
Following A Curve



The Example



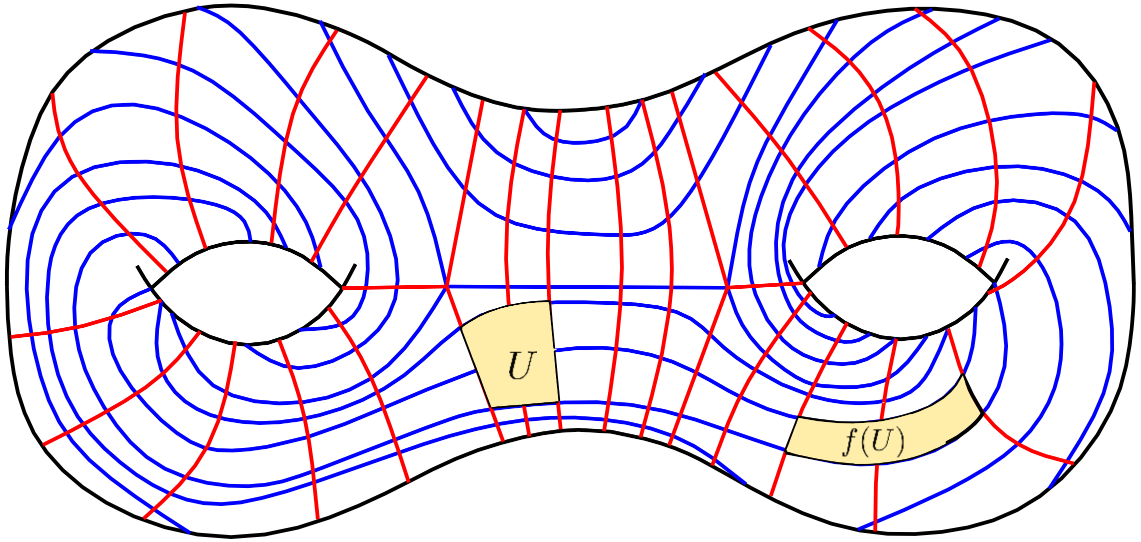
Need a 2-cell attaching map:



$$\ell \simeq f(\ell)$$

A construction of pseudo-Anosov homeomorphisms using positive twists

Yvon Verberne - University of Toronto



Pseudo-Anosov: No power of f maps any curve back to itself

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Prior Constructions:

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New Construction:

Theorem (V.):

Pseudo-Anosov construction using only positive twists

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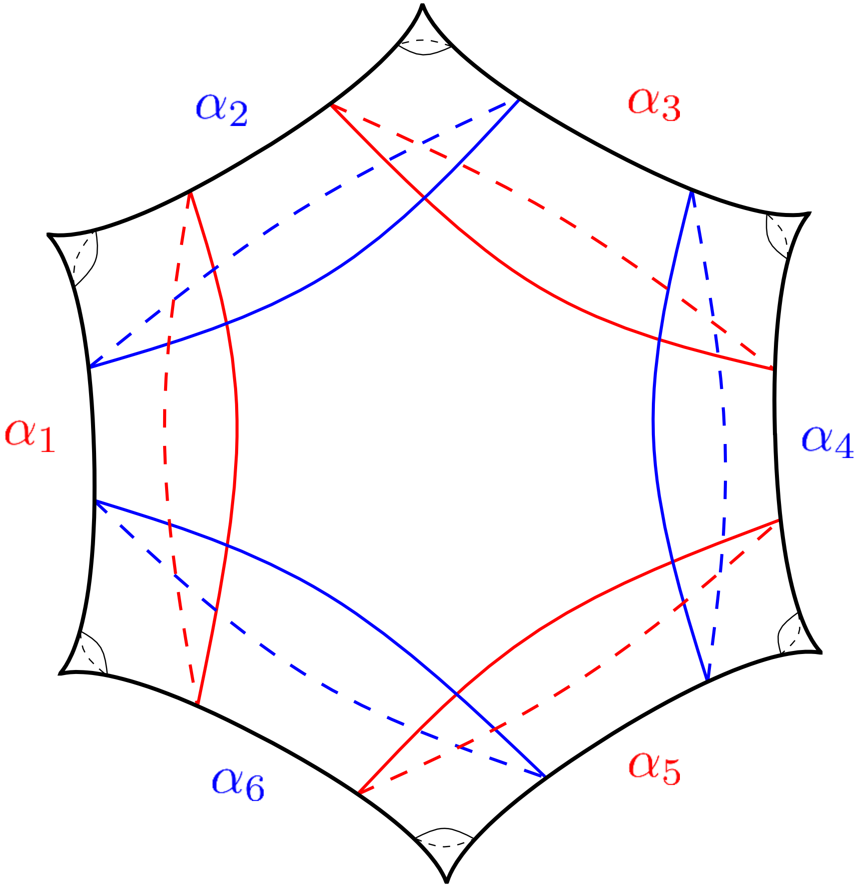

Twist red curves

Twist blue curves

$\rightsquigarrow$ pseudo-Anosov map ϕ

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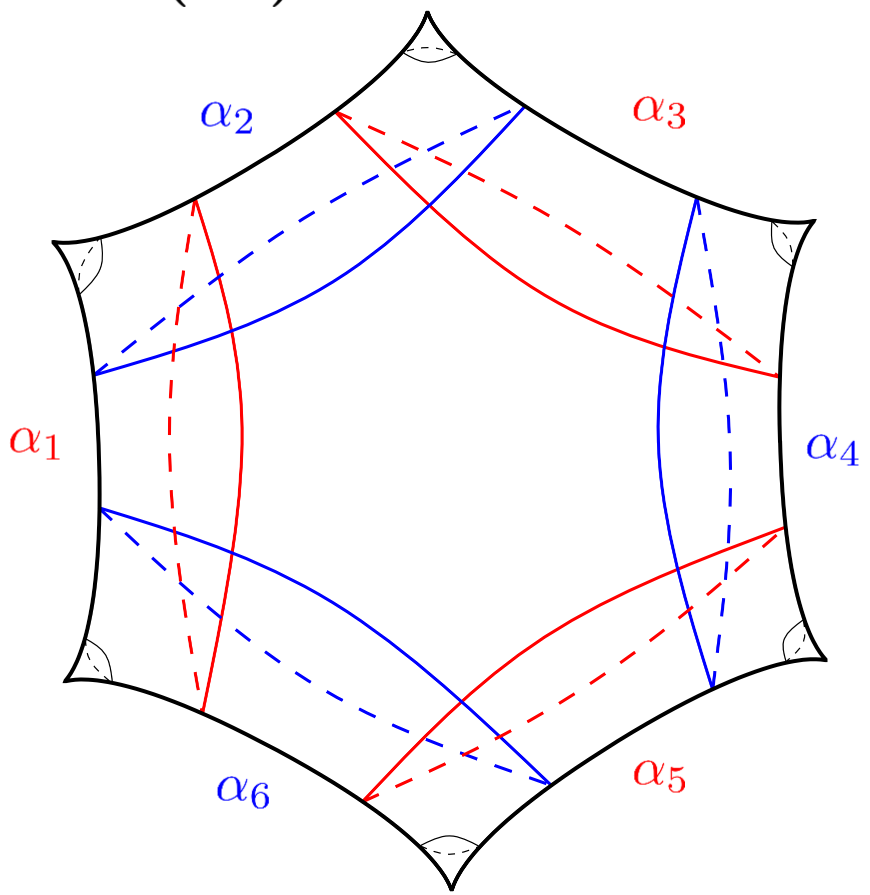
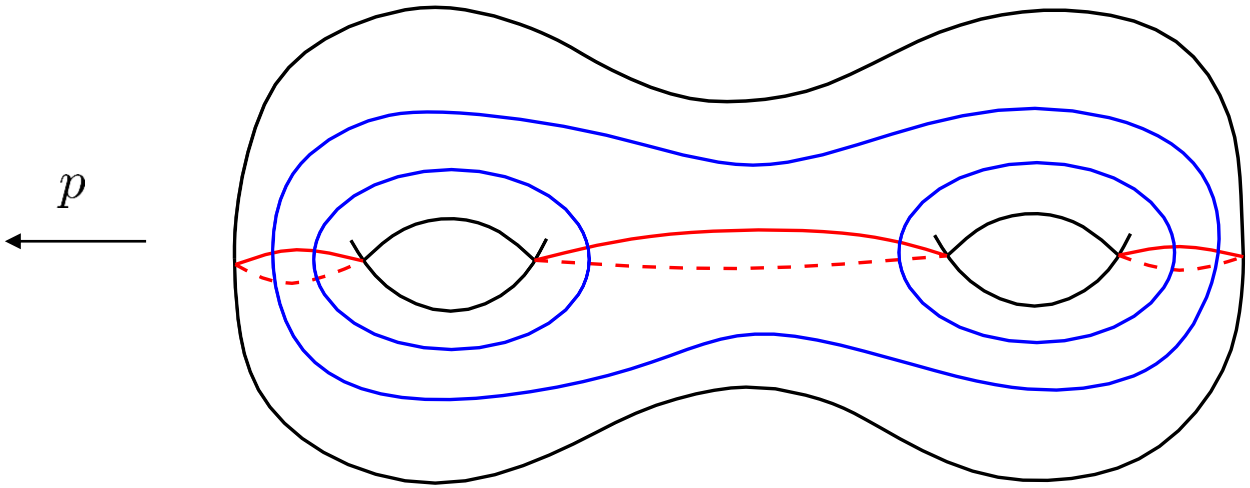
Theorem (V.): Pseudo-Anosov construction using only positive twists





$$\begin{array}{ccc} S_{2,0} & \xrightarrow{\psi} & S_{2,0} \\ \downarrow p & & \downarrow p \\ S_{0,6} & \xrightarrow{\phi} & S_{0,6} \end{array}$$

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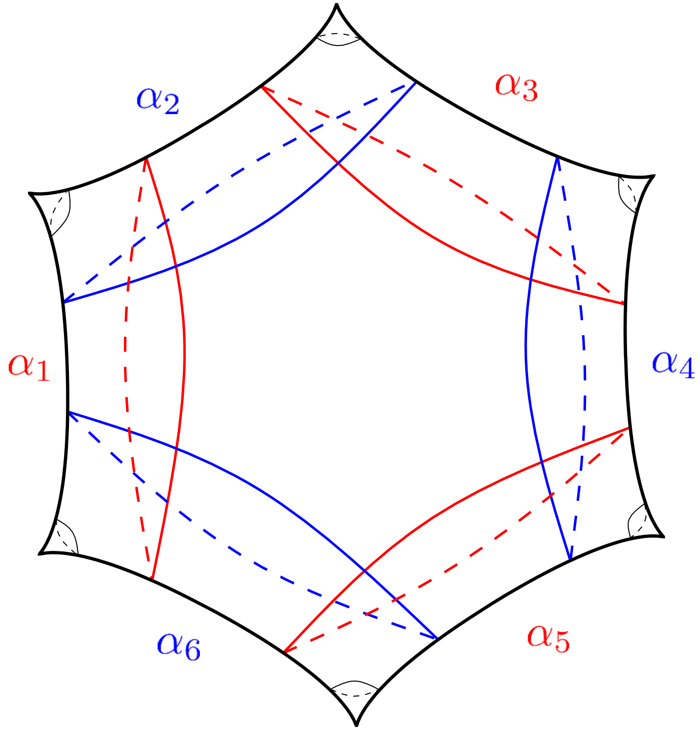

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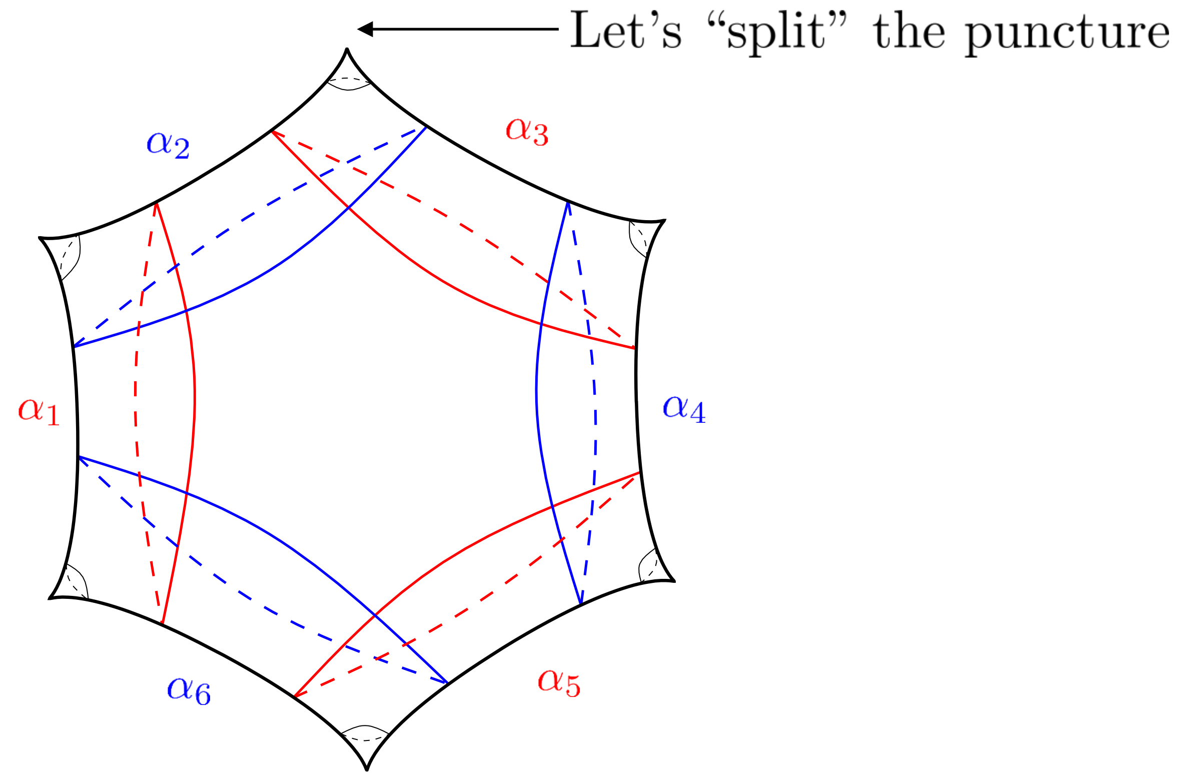

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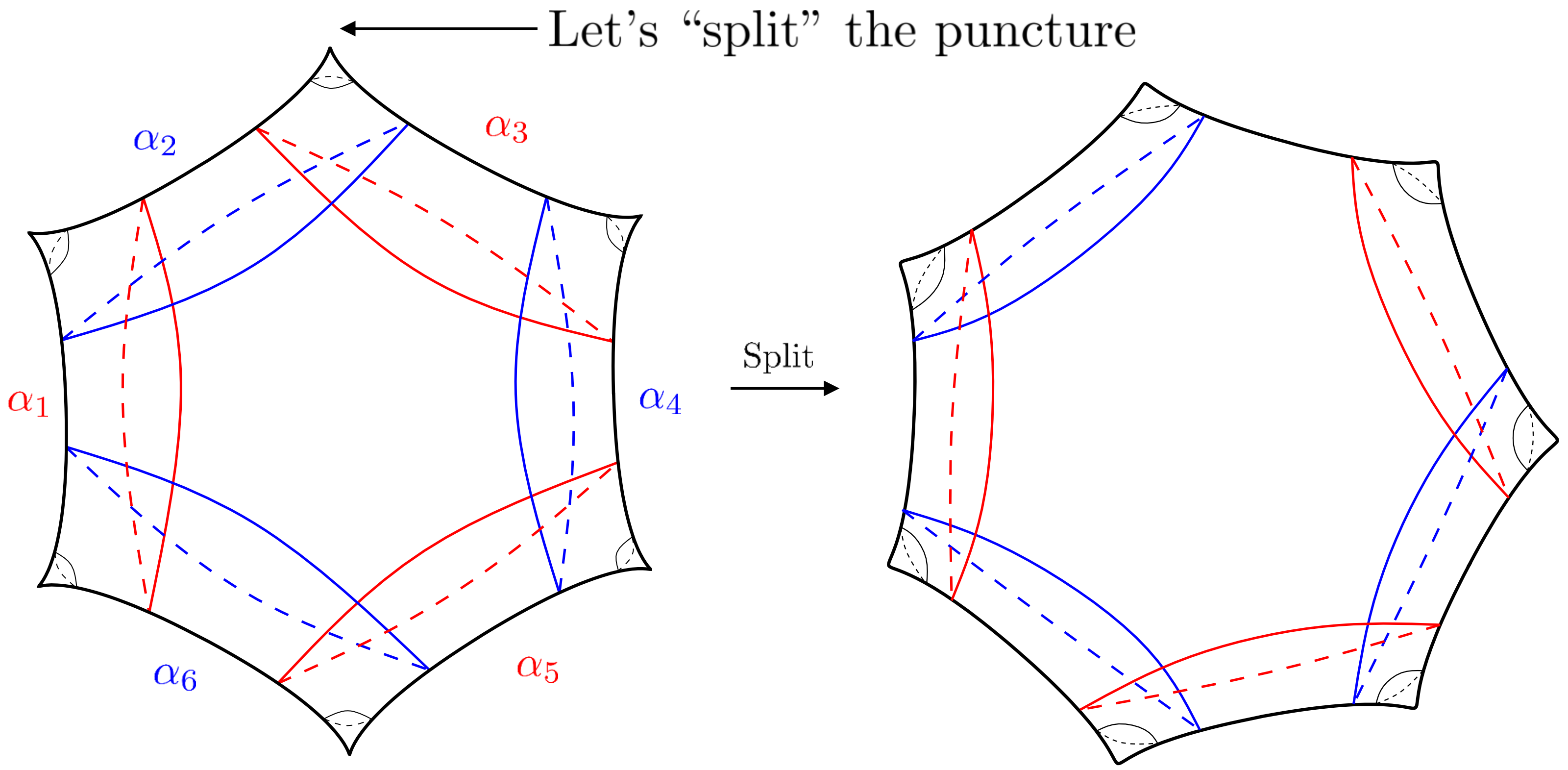

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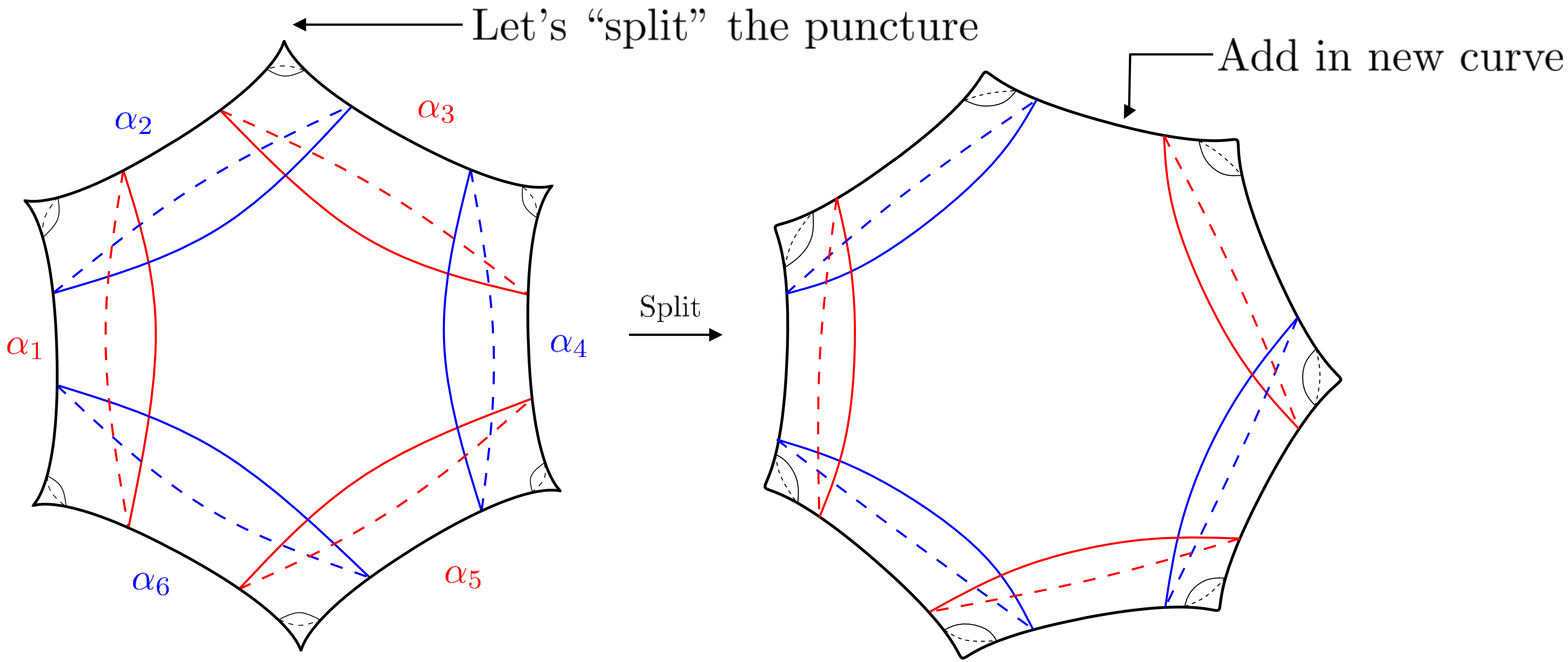

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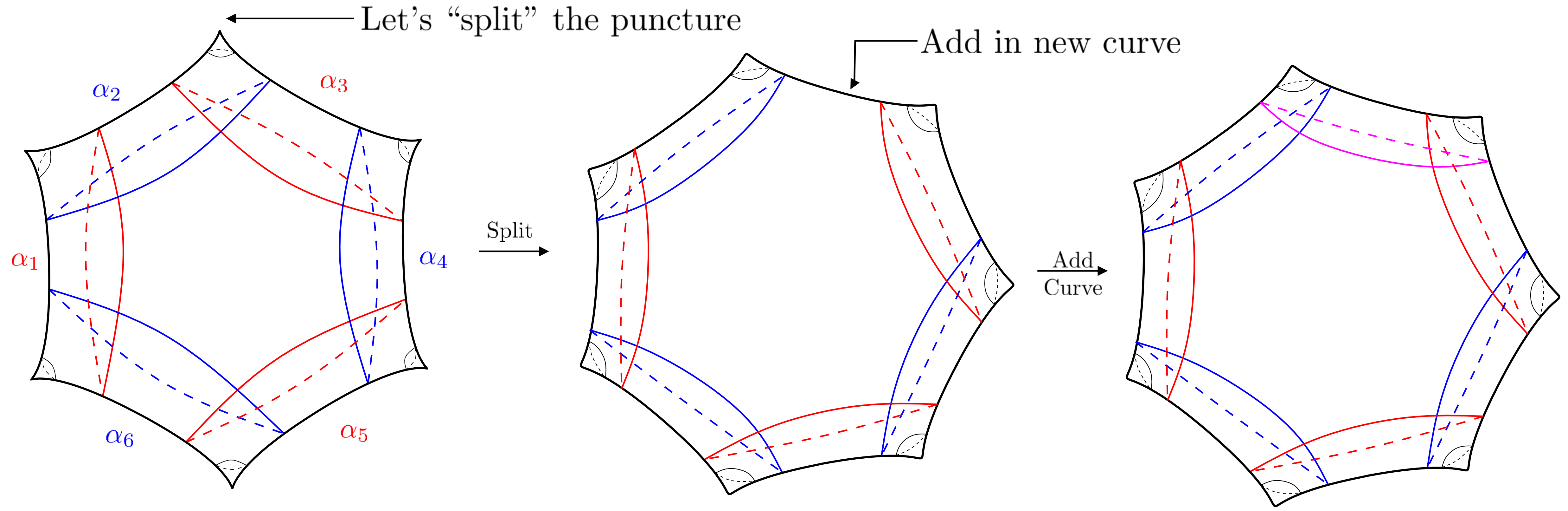

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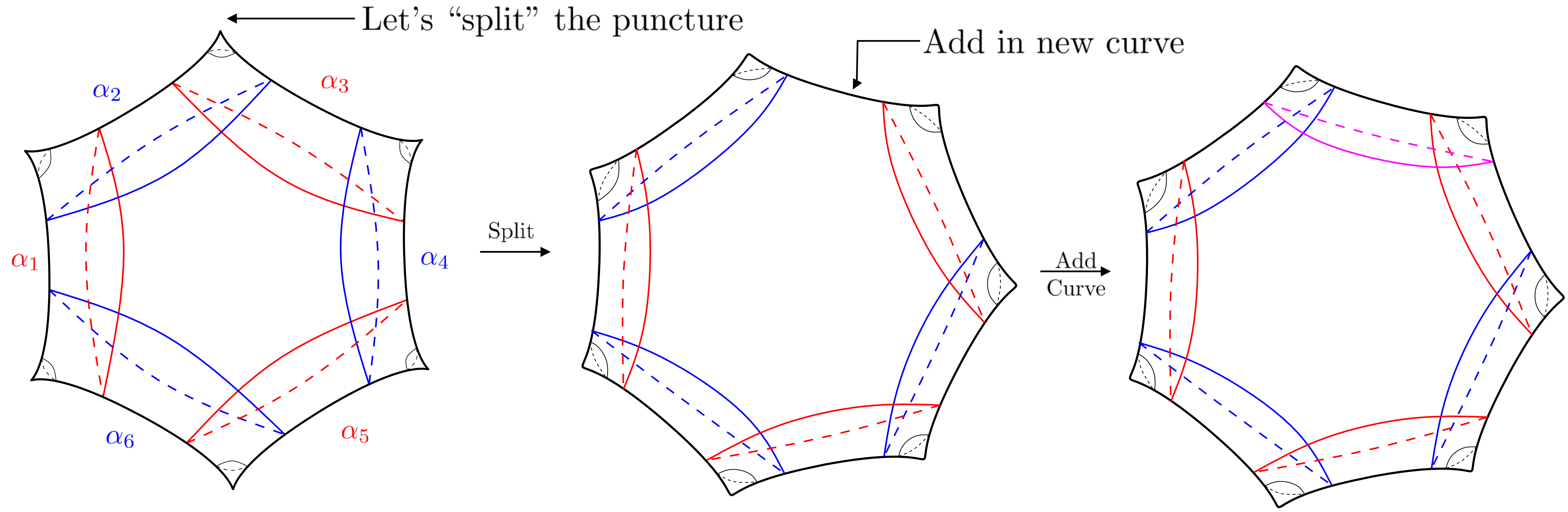

Twist red curves
Twist blue curves
Twist magenta curve
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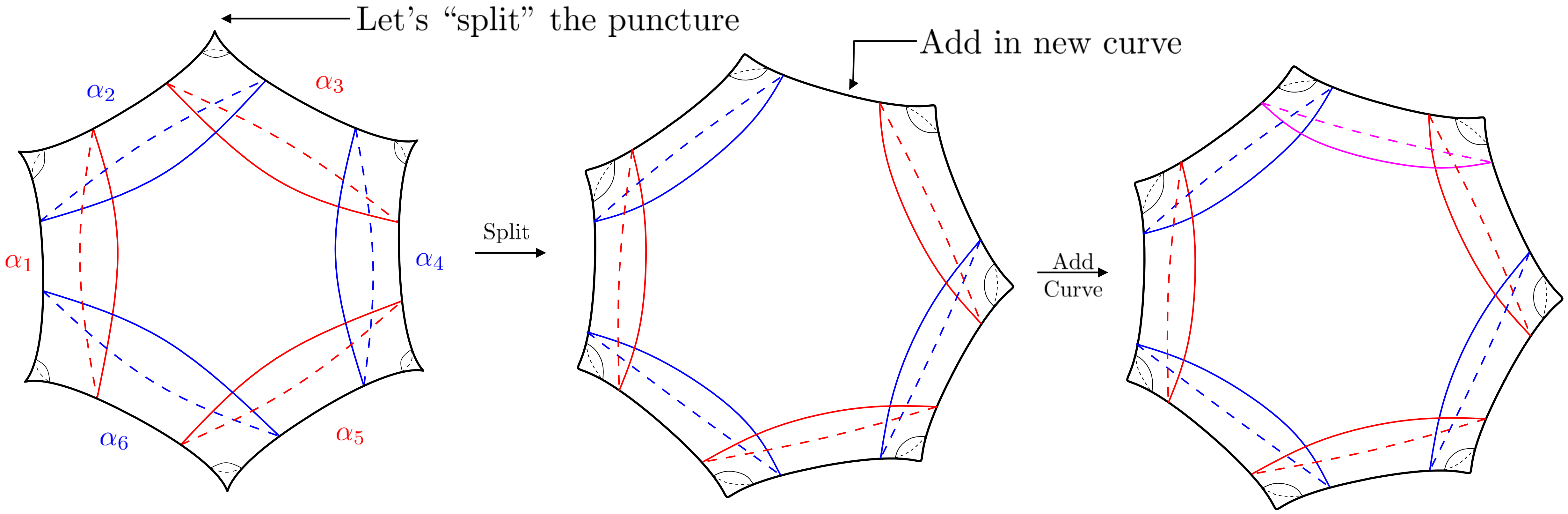

Even coloring plus puncture
splitting $\rightsquigarrow$ pseudo-Anosovs
on punctured spheres

Twist red curves
Twist blue curves
Twist magenta curve
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Theorem (V.): pseudo-Anosov construction using only positive twists

Theorem (V.): this construction provides examples of pseudo-Anosov maps which are unique from both Penner and Thurston's constuctions

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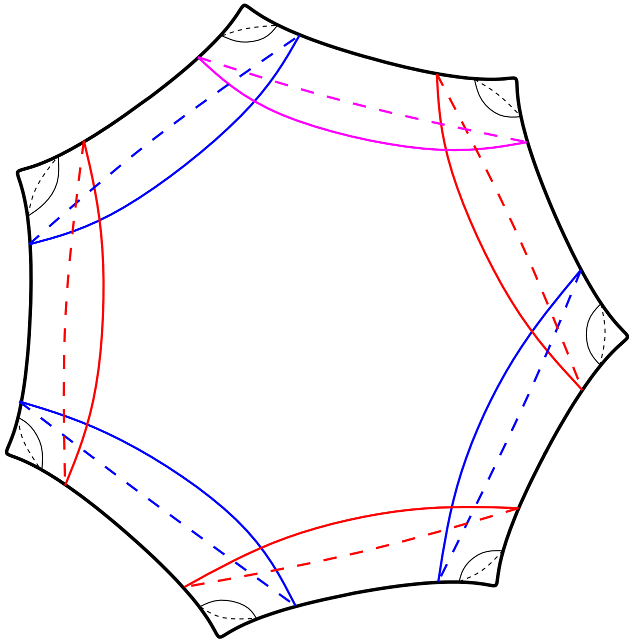

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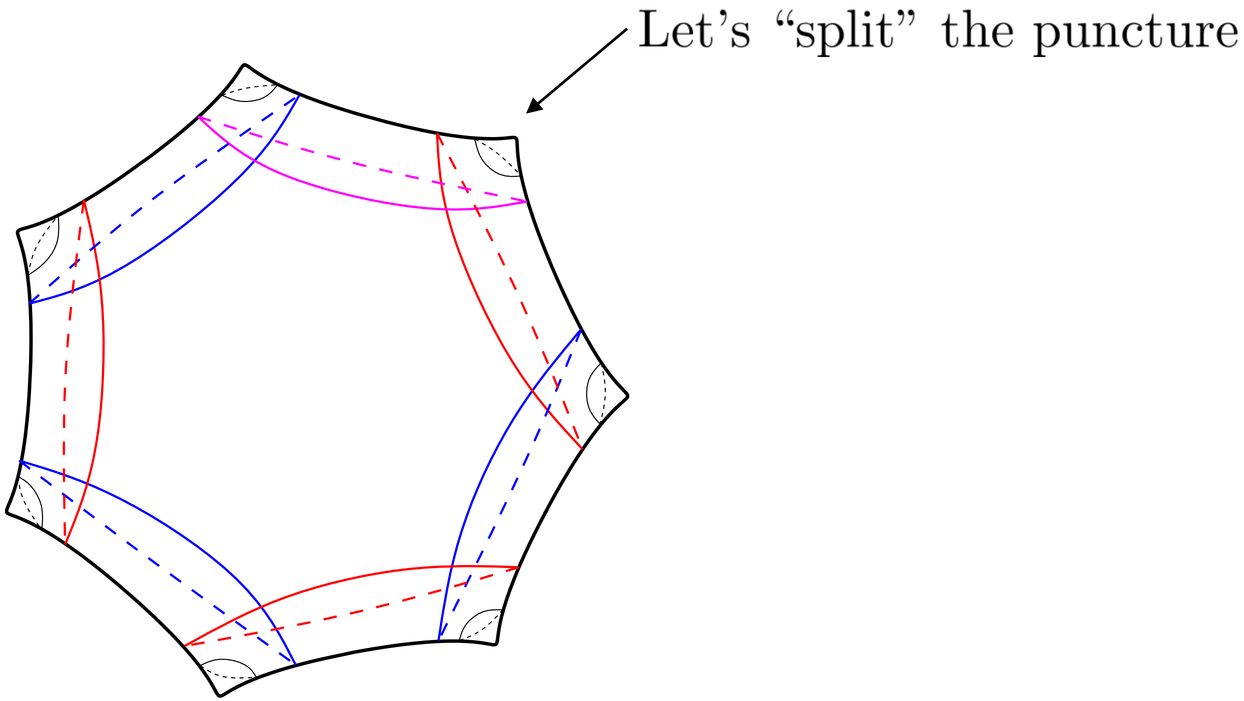

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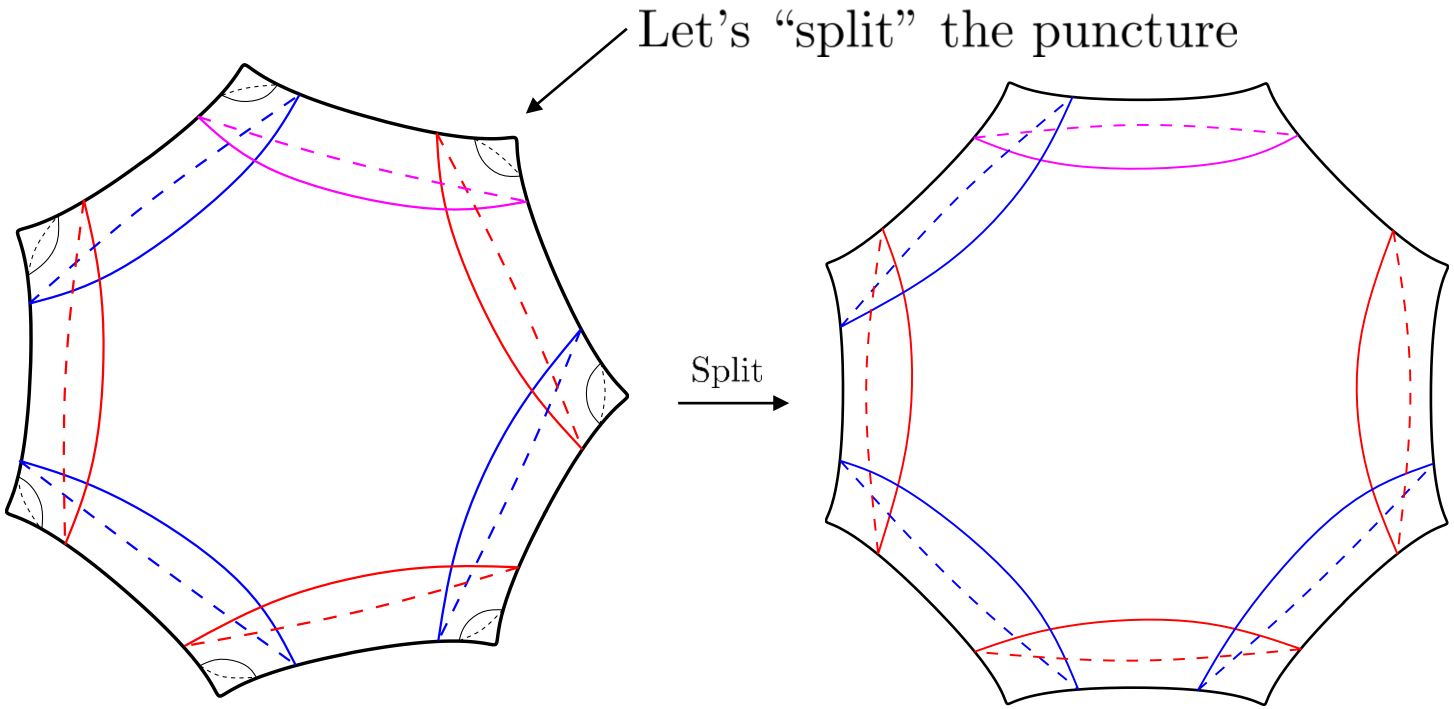

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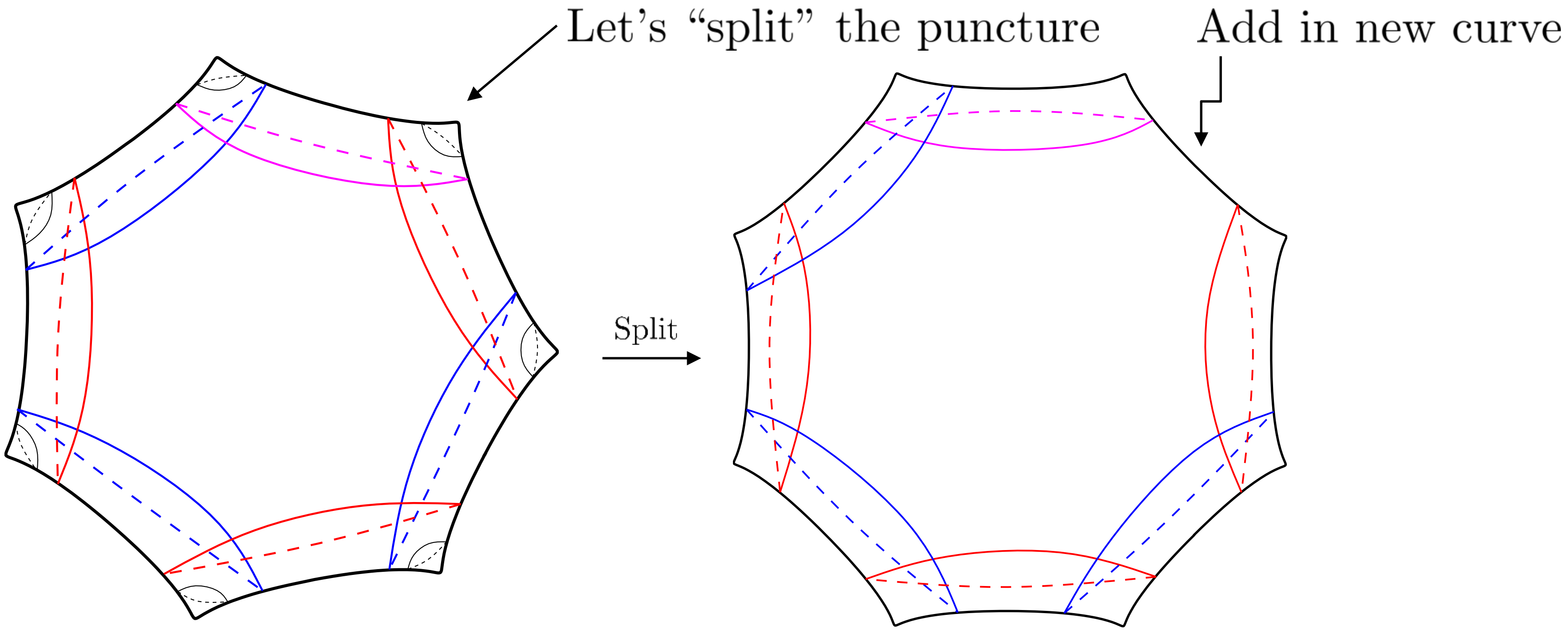

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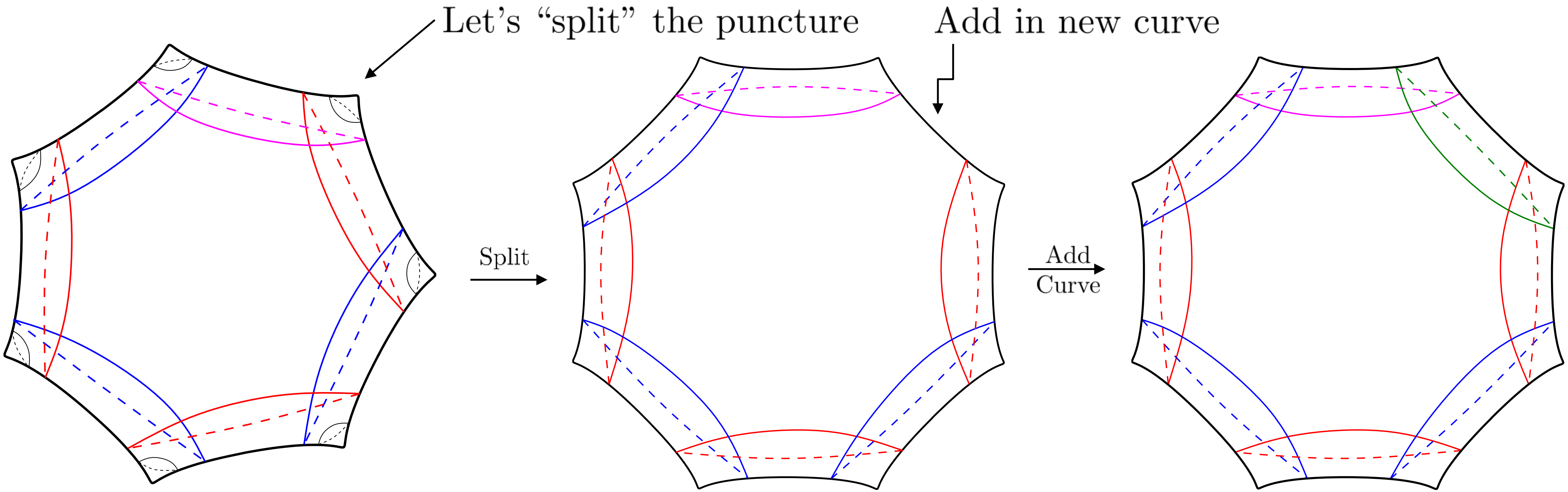

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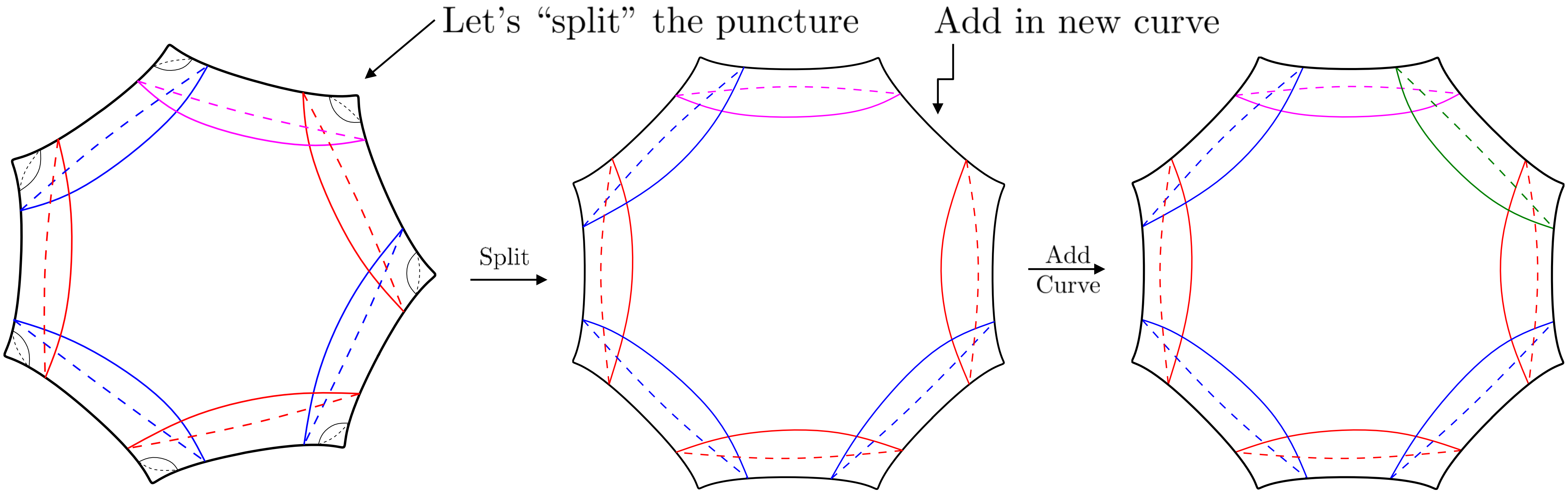


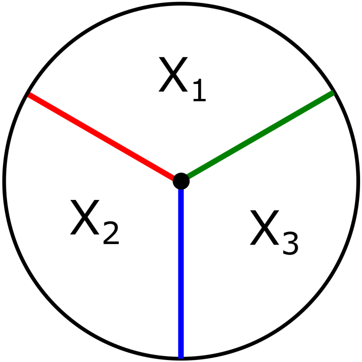
Twist red curves, twist blue curves, twist magenta curve, twist green curve ⇝ pseudo-Anosov map

Diagrams of $\star$-trisections

Román Aranda

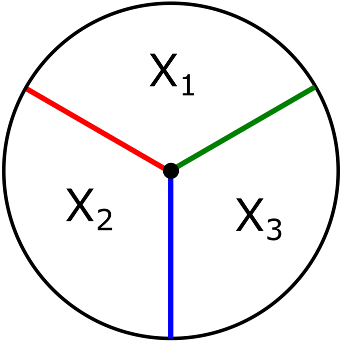
University of Iowa

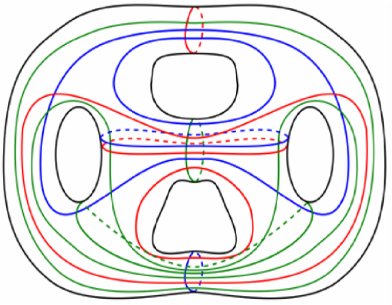
December 2019



Kirby and Gay proved that any smooth 4-manifold has a trisection.

Trisections of 4-manifolds

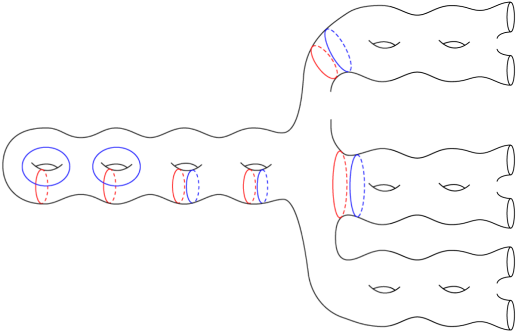




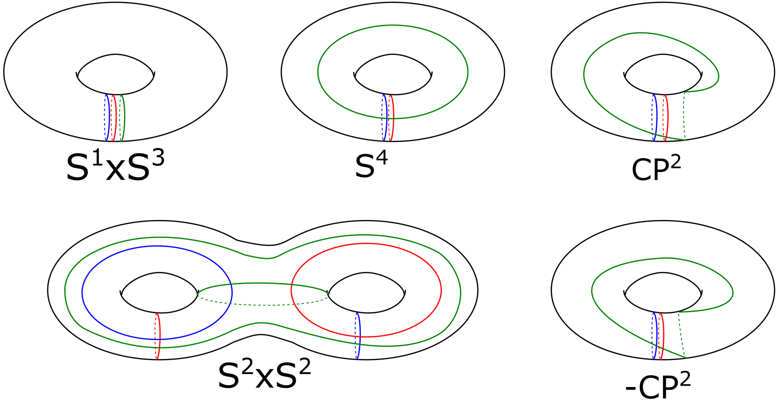
A trisection of X can be decoded using a diagram $(\Sigma; \alpha, \beta, \gamma)$.

Trisection diagrams

Classically, each pair of loops (α, β), (β, γ) and (γ, α) is slide-diffeomorphic equivalent to the standard picture:



Trisection diagrams of small genus



Zupan and Meier proved in 2014 that these are the only irreducible trisections of genus at most two.

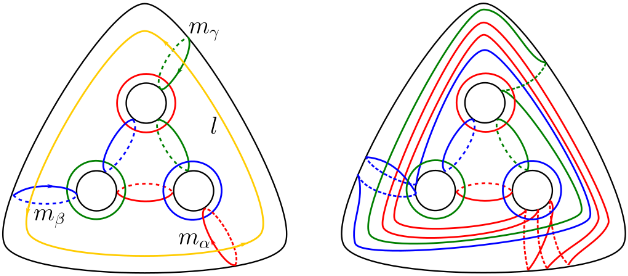
The classification of genus three trisections remains open.

Problems about diagrams

In general, it is not obvious what 4-manifold a given trisection diagram represents.

Farey trisections

Take a triplet of irreducible fractions $\frac{p_i}{q_i} \in \mathbb{Q} \cup \{\frac{1}{0}\}$ satisfying $\det\begin{pmatrix} p_i & p_j \\ q_i & q_j \end{pmatrix} = \pm 1$. Consider the diagram $D(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ as below



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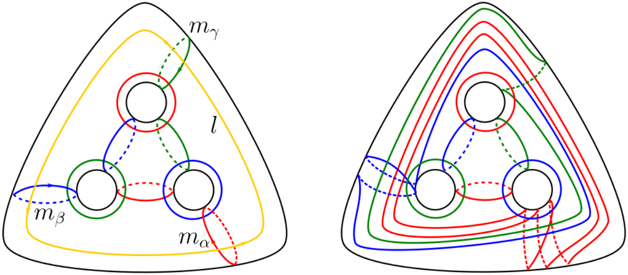

Problem: How many distinct 4-manifolds/trisections are among the diagrams $D(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$?

$\star$-trisection diagrams

In a joint work (Arxiv:1911.06467) with Jesse Moeller, we can loosen the definition of trisection of a 4-manifold to solve the Farey trisections problem using a simple diagramatic perspective.

$\star$-trisection diagrams

Diagram-wise, $\star$-trisections are allowed to have non-isotopic loops of distinct colors which are disjoint from each other.

$\star$-trisection diagrams

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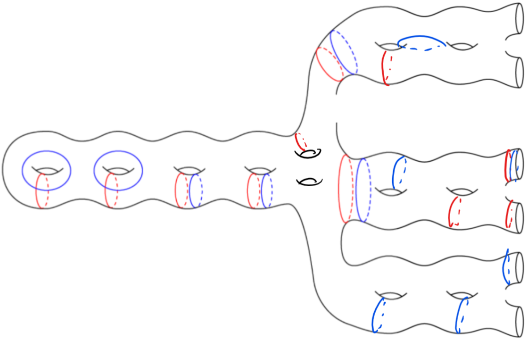
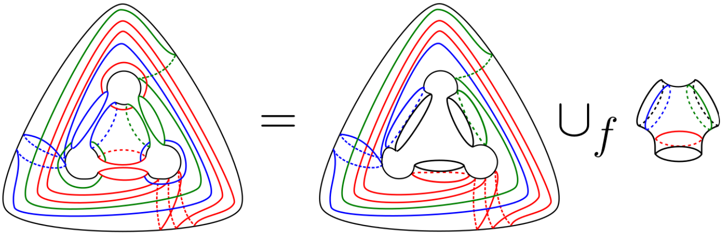
Each pair (α, β), (β, γ) and (γ, α) is slide-diffeomorphic equivalent to the standard picture:



Given a $\star$-trisection diagram $(\Sigma; \alpha, \beta, \gamma)$, the cardinalities $|\alpha|$, $|\beta|$, $|\gamma|$ might not be the same.

Farey trisections

Suppose $\det \left(\begin{smallmatrix} p_i & p_j \\ q_i & q_j \end{smallmatrix}\right) = \pm 1$ for all pairs (i, j).



$$= \left[CP^2 - (\text{loop}) \right] \cup_f S^2 \times D^2$$

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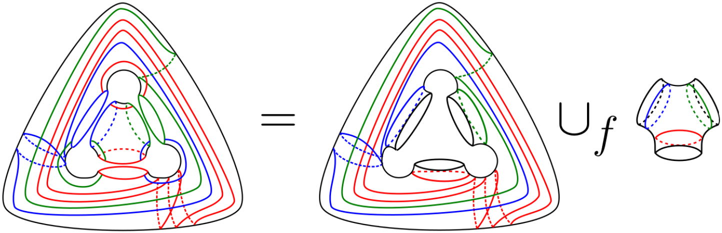

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$$= CP^2 \# (S^2 \times D^2 \cup_f S^2 \times D^2)$$

Farey trisections

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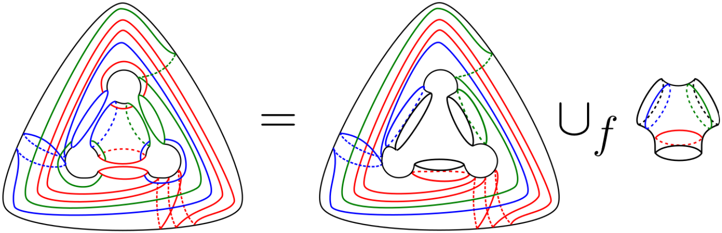

$$= \left[CP^2 - (\text{loop}) \right] \cup_f S^2 \times D^2$$
$$= CP^2 \# (S^2\text{-bundle over } S^2)$$

With a bit more work, we can prove that any two diagrams $D(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ for the same 4-manifold are indeed slide equivalent.

Thank you for your attention!