# Isotopy and equivalence of knots in 3-manifolds

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A knot is an embedding of $S^1$ into $S^3$

$K, J : S^1 \hookrightarrow S^3$:

- <u>isotopic</u>: if $\exists$ 1-parameter family of embeddings $F_t : S^1 \hookrightarrow S^3$ s.t. $F_0 = K$, $F_1 = J$
- <u>ambient isotopic</u>: if $\exists$ a 1-param. family of homeomorphisms $G_t : S^3 \xrightarrow{\sim} S^3$ s.t. $G_0 = Id$ $G_1 \circ K = J$
- <u>equivalent</u>: if $\exists$ a homeomorphism $\phi : S^3 \to S^3$ s.t. $\phi \circ K = J$

$K, J \subset Y$ are isotopic $\Leftarrow$ $K, J \subset Y$ are ambient iso $\Rightarrow$ $K, J \subset Y$ equivalent

isotopy ext. thm

?

Thm [Fisher 60]: Any orientation preserving homeo of $S^3$ is isotopic to Id

Main Thm: Let $Y$ be a closed oriented prime 3-manifold
Every pair of equivalent knots in $Y$ are isotopic $\Leftrightarrow$ $Mod(Y)$ is trivial

Moreover, if an orientation preserving $\phi$ of $Y$ preserves the isotopy class of every knot then $\phi$ is isotopic to Id

Part I:

Th$^{\underline{m}}$ 1: if $\phi$ fixes the conjugacy class of every knot $K$ in $Y$. Then either

① $\phi$ is isotopic to id

② $Y \cong S^1 \times S^2$ and $\phi$ is the Gluck twist

Recall: $Mod(Y) \to Out(\Gamma) \cong Aut(\Gamma)/Inn(\Gamma)$

$$\Gamma = \pi_1(Y)$$

If $Y$ is irreducible, then $Mod(Y) \hookrightarrow Out(\Gamma)$

if $Y \cong S^1 \times S^2$, then $Mod(Y) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$$\downarrow$$

$$Out(\mathbb{Z}) \cong \mathbb{Z}_2$$

Def$^{\underline{n}}$: A group is said to have Grossman's property A if every conjugacy class preserving homomorphism is inner

Th$^{\underline{m}}$ 1': 3-mfd groups have prop. A.

Recall $G$ is called residually finite (RF) if

$$\bigcap_{\substack{H \trianglelefteq G \\ [G:H] < \infty}} H = \{e\}$$

Th$^{\underline{m}}$ (Grossman 74):

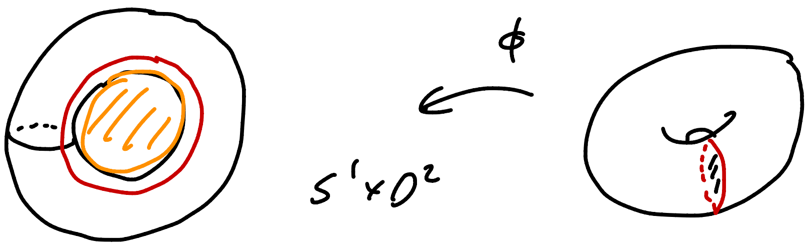
$G$ is finitely gen. conjugacy sep.

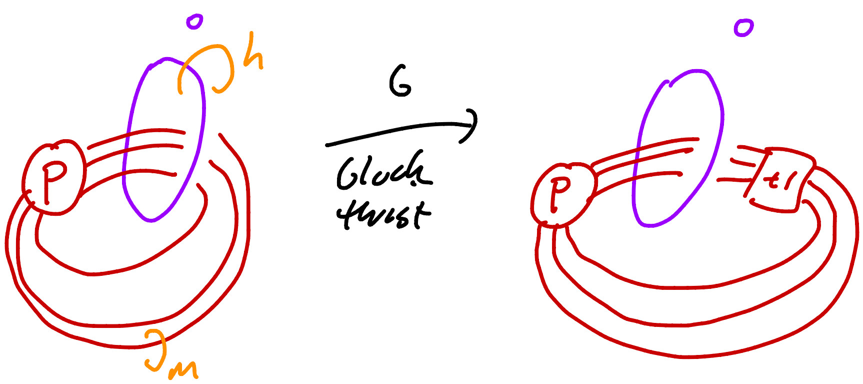
prop A $\Rightarrow$ $Out(G)$ is RF

- Hyp. (and also non-prime) have Prop A [Minayan - Osin '10]
- S.F. w/ base orbifold is not a sphere with 3-cone pts nor torus with 1-cone point [Allenby - Kim - Tang '08, '10] have prop A.

<u>Th^m 1''</u>: the rest have prop A

## <u>Part II</u>: knots in $S^1 \times S^2$




<u>Th^m 2</u>: for each positive integer $w$, there exist a winding number $w$ knot $K$ in $S^1 \times S^2$ s.t.
$K \not\cong_{iso} G(K)$
Moreover, if $w > 1$ and odd, then $K \not\cong_{iso} G(K)$

<u>Rmk</u>: $K \cong_{iso} G(K)$

consider parallel copy of $K$ (w/o linking) $\lambda$

note $lk(G(K), G(\lambda)) = \omega^2$

<u>lemma</u>: Suppose $K \cong_{iso} G(K)$ then there exist a homeo

$$\phi : S^1 \times S^2_1(K) \xrightarrow{\sim} S^1 \times S^2_{1+\omega^2+2k\omega}(K)$$

for some $k$

Moreover, $\phi_*([h]) = [h] + (\omega+k)[m]$

① Casson-Gordon Sig

If $\omega > 1$ then $\omega^2 + 2k\omega = 0$

$\Rightarrow \omega = -2k$

$\therefore$ winding # must be <u>odd</u>

② d-invt

$$S^1 \times S^2_1(K) \xrightarrow[\phi]{\sim} S^1 \times S^2_1(G(K))$$

$$\phi_*([h]) = [h] + \tfrac{1}{2}\omega[m]$$