Homology Cobordism & Involutive Heegaard Floer Homology

w/ J. Hom (Georgia Tech), M. Stoffregen (MSU)
& I. Zemke (Princeton)

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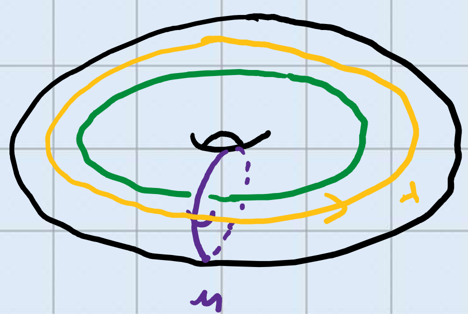
Background on $\Theta^3_{\mathbb{Z}}$ –

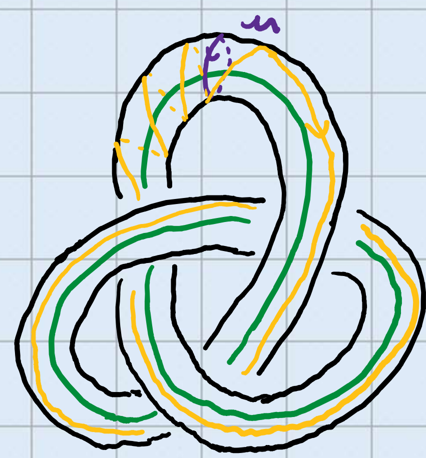
Defn An integer homology sphere, or $\mathbb{Z}HS^3$, is an oriented 3-mfd Y w/ $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$.

Two integer homology spheres are homology cobordant, $Y_1 \sim Y_2$, if there is a smooth cpt oriented W^4 st $\partial W^4 = -Y_1 \amalg Y_2$ and $H_*(Y_i;\mathbb{Z}) \xrightarrow{\sim} H_*(W^4;\mathbb{Z})$

Some examples

- For $K \subseteq S^3$, the surgery $S^3_{\pm 1/q}(K)$





cut out a solid torus along K and reglue so that μ goes to $p\mu + q\lambda$
$\rightsquigarrow S^3_{p/q}(K)$

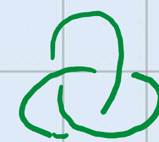
- For $(p,q,r)=1$, $\Sigma(p,q,r) = \{ z_1^p + z_2^q + z_3^r = 0 \} \cap S^5_\varepsilon \subseteq \mathbb{C}^3$

<u>Brieskorn sphere</u>

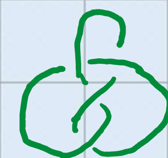
Examples of the Examples

- $\Sigma(2,3,5) = -S^3_{+1}(3_1)$ is the Poincaré homology sphere.

- $\Sigma(2,3,7) = S^3_{-1}(3_1) = S^3_{+1}(4_1)$



$3_1 = T_{2,3}$

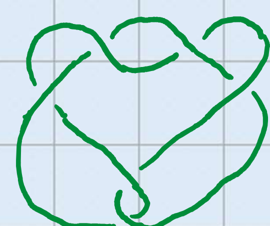


4_1

- Some null homology cobordisms:

- If $K \subseteq S^3$ slice (bounds a smooth disk in B^4), $S^3_{\pm 1/q}(K) \sim S^3$

eg



- (Mazur, Kirby-Akbulut) $\Sigma(2,5,7) \sim S^3$ ∂

The group $\Theta^3_{\mathbb{Z}}$ —

Defn The integer homology cobordism group is

$$\Theta^3_{\mathbb{Z}} = \left(\{ Y \text{ an oriented } \mathbb{Z}HS^3 \}, \# \right) / \sim$$

- $[S^3]$ is the identity; $[Y] = [S^3] \Leftrightarrow Y$ bounds a smooth $\mathbb{Z}HB^4$.

- $-[Y] = [-Y]$, since $Y \# -Y \sim S^3$ along $(Y \times I) - B^4$

Smoothness matters (Freedman) Every $\mathbb{Z}HS^3$ bounds an acyclic **topological** 4-ball.

Dimension matters $\Theta^1_{\mathbb{Z}} = \Theta^2_{\mathbb{Z}} = 0$; $\Theta^n_{PL} = 0$ for $n \geq 4$ (Kervaire '63)

What do we know about $\Theta^3_{\mathbb{Z}}$?

- Until the 1980s, mostly the Milnor-Rokhlin homomorphism

$$\mu : \Theta^3_{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$$
$$\Sigma(2,3,5) \longmapsto 1$$

Pick W^4 spin w/ $\partial W^4 = Y$; $\mu(Y) = \frac{1}{8}\sigma(W^4)$.

- (Fintushel-Stern '85, '90), (Furuta '90)

$$\mathbb{Z} \hookrightarrow \Theta^3_{\mathbb{Z}}$$
$$n \longmapsto [\Sigma(2,3,5)]^n$$

$$\begin{tikzcd} \mathbb{Z}^\infty \arrow[r, hook] \arrow[dr] & \Theta^3_{\mathbb{Z}} \\ & \Theta^3_{SF} \arrow[u, hook] \end{tikzcd}$$

Θ^3_{SF} } Subgroup gen'd by Seifert Fibred spaces

- (Frøyshov '10) $\Theta^3_{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}$, so $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}$ summand.

$$[\Sigma(2,3,5)] \longmapsto 1$$

<u>ctd...</u>

- (Manolescu '13) $\nexists$ $[Y]$ of order 2 w/ $\mu(Y) = 1$. (*)

  - <u>Why important?</u> Work of Galewski-Stern & Matumoto from the 70s shows (*) $\Leftrightarrow$ $\exists$ nontriangulable mfds in every dimn ≥ 5.

<u>Remark</u> Existence of torsion in general is open.

<u>Thm</u> (Dai-Hom-Stoffregen-Truong '18) $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ summand.

Can also ask questions about which classes are represented by which types of 3-mfds...

- Is every class in $\Theta^3_{\mathbb{Z}}$ represented by an irreducible mfd?
    - Yes, Livingston '81

- Is every class in $\Theta^3_{\mathbb{Z}}$ represented by a hyperbolic mfd?
    - Yes, Myers '83

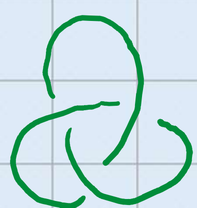
- Is every class in $\Theta^3_{\mathbb{Z}}$ represented by a SFS?
    - No, Stoffregen '15, Lin '15, Frøyshov

- Is every class in $\Theta^3_{\mathbb{Z}}$ represented by a surgery on a knot?
    - No, Nozaki-Sato-Taniguchi '19

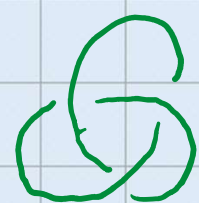
- Is every class in $\Theta^3_{\mathbb{Z}}$ represented by a Stein fillable 3-mfd?
    - Yes, Mukherjee '20

• Is $\Theta^3_{\mathbb{Z}} = \Theta^3_{SF}$?

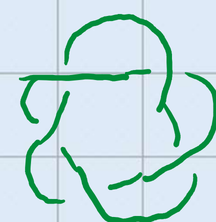
<u>Thm</u> (H-Hom-Stoffregen-Zemke) No. Indeed the classes $[S^3_{+1}(T_{2,3} \# -2T_{2n,2n+1} \# T_{2n,4n+1})]$ For $n \geq 3$, n odd generate a subgroup $\mathbb{Z}^\infty \leq \Theta^3_{\mathbb{Z}} / \Theta^3_{SF}$.



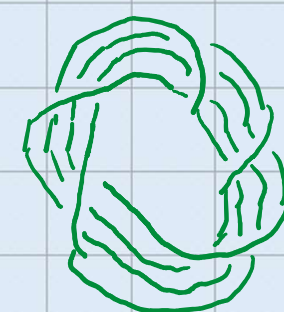
$T_{2,3}$



$-T_{2,3}$



$T_{2,5}$



$T_{4,5}$

Where does this come from?

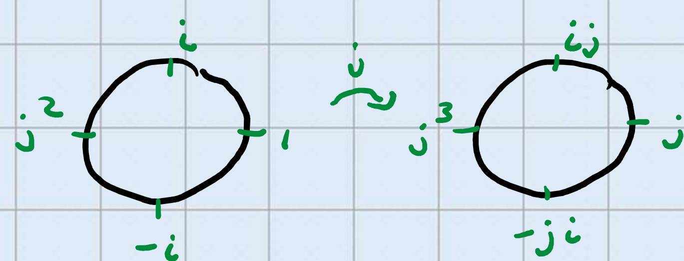
- Manolescu's disproof of the Triangulation Conjecture used a $Pin(2)$-equivariant version of his Seiberg-Witten Floer homology

Morally

$Y \rightsquigarrow$ $Pin(2)$ spectrum constructed

a $\mathbb{Z}HS^3$ From solns to the Seiberg-Witten eqns

$Pin(2)$ subgroup of the unit quaternions



$j^2 = -1$

$ij = -ji$

- S^1-equivariant SWFH $\Leftrightarrow$ various versions of Ozsváth-Szabó's Heegaard Floer homology

(Lidman-Manolescu, Kutluhan-Lee-Taubes, Colin-Ghiggini-Honda + Taubes)

(Involutive) Heegaard Floer Homology

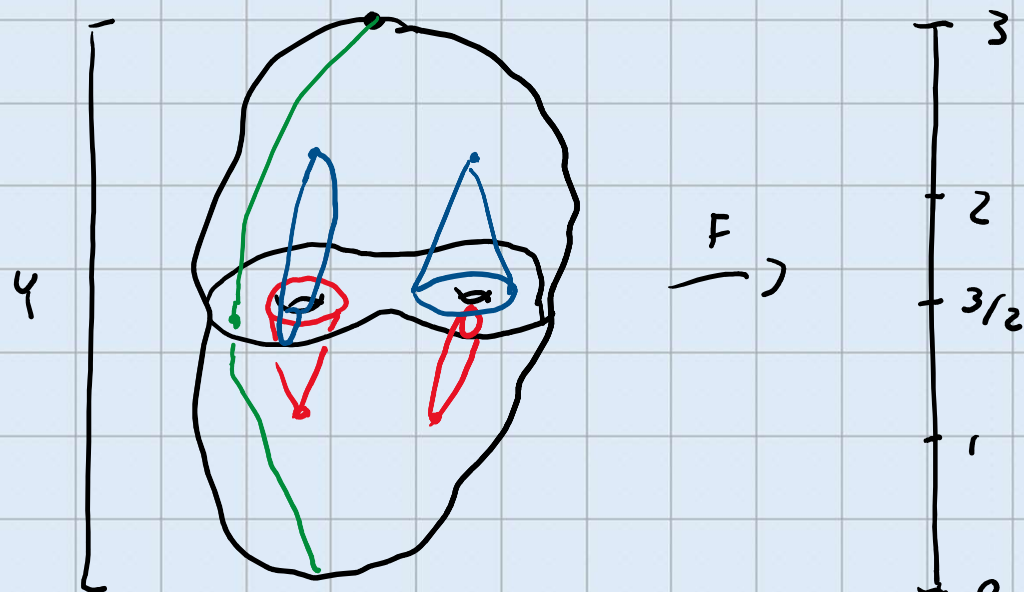
- Y a $\mathbb{Z}HS^3 \rightsquigarrow (CF^-(Y), \iota_Y)$

Ozsváth-Szabó early 2000s

H-Manolescu '15,
H-Manolescu-Zemke '16

- $CF^-(Y)$ free finitely-gen'd graded $\mathbb{F}_2[U]$ complex, $\deg(U) = -2$
- $U^{-1}H_*(CF^-(Y)) \simeq \mathbb{F}_2[U, U^{-1}]$
- $\iota_Y : CF^-(Y) \longrightarrow CF^-(Y)$; $\iota_Y^2 \simeq Id$.

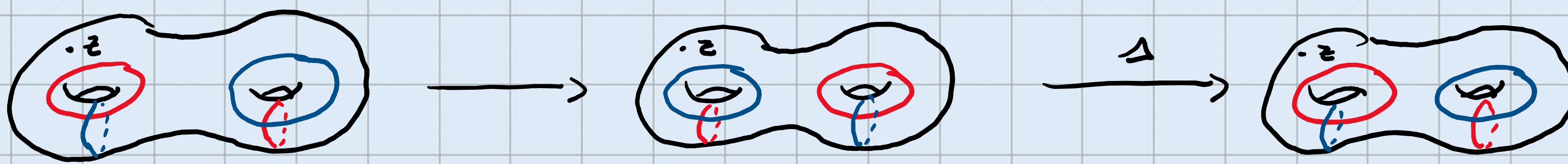
Construction



$\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$

- Generators are tuples of intersections of curves
- ∂ counts solns to pdes in an auxiliary symplectic mfd

Construction of ι_Y



$\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$ $\qquad$ $\bar{\mathcal{H}} = (-\Sigma, \vec{\beta}, \vec{\alpha}, z)$ $\qquad$ $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$

$$\iota_Y : CF^-(\mathcal{H}) \xrightarrow[\eta]{\sim} CF^-(\bar{\mathcal{H}}) \xrightarrow[\Phi(\bar{\mathcal{H}}, \mathcal{H})]{} CF^-(\mathcal{H})$$

che

- $Y_1 \# Y_2 \rightsquigarrow \left(CF^-(Y_1) \otimes_{\mathbb{F}_2[U]} CF^-(Y_2),\ \iota_{Y_1} \otimes \iota_{Y_2} \right)$

Several notions of equivalence

0 Equivariant che

$$CF^-(Y_1) \underset{g}{\overset{f}{\rightleftarrows}} CF^-(Y_2)$$

$$f\iota_1 \simeq \iota_2 f \qquad g\iota_2 \simeq \iota_1 g$$

$$\rightsquigarrow HFI^-(Y) = H_*\left(\mathrm{Cone}\left(CF^-(Y) \xrightarrow{Q(1+\iota_Y)} Q\,CF^-(Y)\right)\right)$$

module ove $\mathbb{F}_2[U, Q]/(Q^2)$ $\quad \deg Q = -1$

$\rightsquigarrow$ <u>not</u> an invt of homology cobordism.

1 Local Equivalence

Defn An iota-complex (C, ι) is a free finitely gen'd $\mathbb{F}_2[U]$-cpx with $U^{-1}H_* \simeq \mathbb{F}_2[U, U^{-1}]$ and a grading preserving map ι st

$$\partial\iota + \iota\partial = 0 \qquad \iota^2 + Id + \partial H + H\partial = 0$$

Two iota-complexes are locally equivalent if there are maps

$$C_1 \underset{g}{\overset{f}{\rightleftarrows}} C_2$$

st f_*, g_* are isomorphisms on $U^{-1}H_*$ and

$$F\iota_1 + \iota_2 F + \partial F + F\partial = 0 \qquad g\iota_2 + \iota_1 g + \partial G + G\partial = 0.$$

Induces

$$\Theta^3_{\mathbb{Z}} \longrightarrow \underbrace{\mathfrak{I}}_{\substack{\text{Group of} \\ \text{iota-complexes} \\ \text{up to l.e.} \\ \text{w/ } \otimes}}$$

1/2 Almost Local Equivalence

Dai-Hom-Stoffregen-Truong

Defn An almost iota-complex $(C, \bar\iota)$ is a free finitely gen'd $\mathbb{F}_2[U]$-cpx with $U^{-1}H_* \simeq \mathbb{F}_2[U, U^{-1}]$ and a grading preserving map $\bar\iota$ st

$$\partial\bar\iota + \bar\iota\partial = \cancel{0} \in \mathrm{Im}(U) \qquad \bar\iota^2 + Id + \partial H + H\partial = \cancel{0} \in \mathrm{Im}(U)$$

Two almost iota-complexes are almost locally equivalent if there are maps

$$\begin{tikzcd} C_1 \arrow[r, shift left, "f"] & C_2 \arrow[l, shift left, "g"] \end{tikzcd}$$

st f_*, g_* are isomorphisms on $U^{-1}H_*$ and

$$F\bar\iota_1 + \bar\iota_2 F + \partial F + F\partial = \cancel{0} \in \mathrm{Im}(U) \qquad g\bar\iota_2 + \bar\iota_1 g + \partial G + G\partial = \cancel{0} \in \mathrm{Im}(U).$$

Induces

$$\begin{tikzcd} \Theta^3_{\mathbb{Z}} \arrow[r] & \mathcal{I} \arrow[r] & \overline{\mathcal{I}} \\ \Theta^3_{SF} \arrow[u, phantom, "\cup" description] \arrow[urr, bend right] & & \end{tikzcd}$$

$\underbrace{\mathcal{I}}$: Group of iota-complexes up to l.e. w/ $\otimes$

$\underbrace{\overline{\mathcal{I}}}$: Group of almost iota-complexes up to a.l.e. w/ $\otimes$

Thm (Dai-Hom-Stoffregen-Truong) $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$-summand.

Thm (=) The almost iota complex $C(n)$ for $n \geq 2$

$$\alpha \xleftarrow{1+\bar{\iota}} \beta \xrightarrow{U} \gamma \xleftarrow{1+\bar{\iota}} \delta \xrightarrow{U^n} \varepsilon$$

$$\partial\beta = U\gamma \qquad \partial\delta = U^n\varepsilon$$

$$\bar{\iota}(\beta) = \alpha + \beta$$

$$\bar{\iota}(\delta) = \gamma + \delta$$

Cor (HHSZ) Indeed, neither is any sum

$$C = \pm C(n_1) \pm C(n_2) \pm \cdots \pm C(n_m).$$

<u>Thm</u> (HHSZ) There exists a formula to compute the involutive Floer homology of a surgery $S^3_{p/q}(K)$ from involutive invts associated to the knot.

(... which are themselves computable for torus knots (H-Manolescu) and connect sums (Zemke) ...)

<u>Thm (HHSZ)</u> almost local equivalence class of

$$S^3_{+1}(T_{2,3} \# -2T_{2n,2n+1} \# T_{2n,4n+1}) \text{ is } C(n-1) \text{ for } n \geq 3 \text{ odd.}$$

$\rightsquigarrow$ main thm.

Thanks for your

time!

Bonus Pictures

$CF^-(S^3)$ $\quad$ $HF^-(S^3) = \mathbb{F}_{(0)}[U]$

$$\begin{array}{c} x \\ \\ Ux \\ \\ U^2x \\ \vdots \end{array}$$

$CF^-(\Sigma(2,3,7))$ $\quad$ $HF^-(\Sigma(2,3,7)) = \mathbb{F}_{(0)}[U] \oplus \mathbb{F}_{(0)}$

$$\begin{tikzcd}[column sep=small]
a & & b \\
& c \arrow[dl] \arrow[dr] & \\
Ua & & Ub \\
& Uc \arrow[dl] \arrow[dr] & \\
U^2a & & U^2b \\
& \vdots &
\end{tikzcd}$$

$$F : CF^-(\Sigma(2,3,7)) \longrightarrow CF^-(S^3)$$

$$a \longmapsto x$$

$$b \longmapsto x$$

$$c \longmapsto 0$$

$$\nexists \; g : CF^-(S^3) \longrightarrow \Sigma(2,3,7)$$

eg if $g(x) = a$, $\left.\begin{array}{l} \iota g(x) = b \\ g\iota(x) = a \end{array}\right\} \times$

A local equivalence $\quad Y = \Sigma(2,3,7)$

$CF^-(Y \# Y)$ $\qquad\qquad$ $HF^-(Y \# Y) \simeq \mathbb{F}_{(0)}[U] \oplus \mathbb{F}_{(0)}^{\oplus 3} \oplus \mathbb{F}_{(-1)}$

$$a_1a_2 \qquad b_1a_2 \qquad a_1b_2 \qquad b_1b_2$$

$$\begin{tikzcd}
c_1a_2 \arrow[d] \arrow[dr] & a_1c_2 \arrow[dl] \arrow[dr] & b_1c_2 \arrow[dl] \arrow[dr] & c_1b_2 \arrow[dl] \arrow[d] \\
Ua_1a_2 & Ub_1a_2 & Ua_1b_2 & Ub_1b_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & & c_1c_2 \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\
Uc_1a_2 \arrow[d] \arrow[dr] & Ua_1c_2 \arrow[dl] \arrow[dr] & Ub_1c_2 \arrow[dl] \arrow[dr] & Uc_1b_2 \arrow[dl] \arrow[d] \\
U^2a_1a_2 & U^2b_1a_2 & U^2a_1b_2 & U^2b_1b_2
\end{tikzcd}$$

A local equivalence $Y = \Sigma(2,3,7)$

$CF^-(Y \# Y)$

$H_*(C) \simeq \mathbb{F}_{(0)}[U] \oplus \mathbb{F}_{(0)} \oplus \mathbb{F}_{(-1)}$

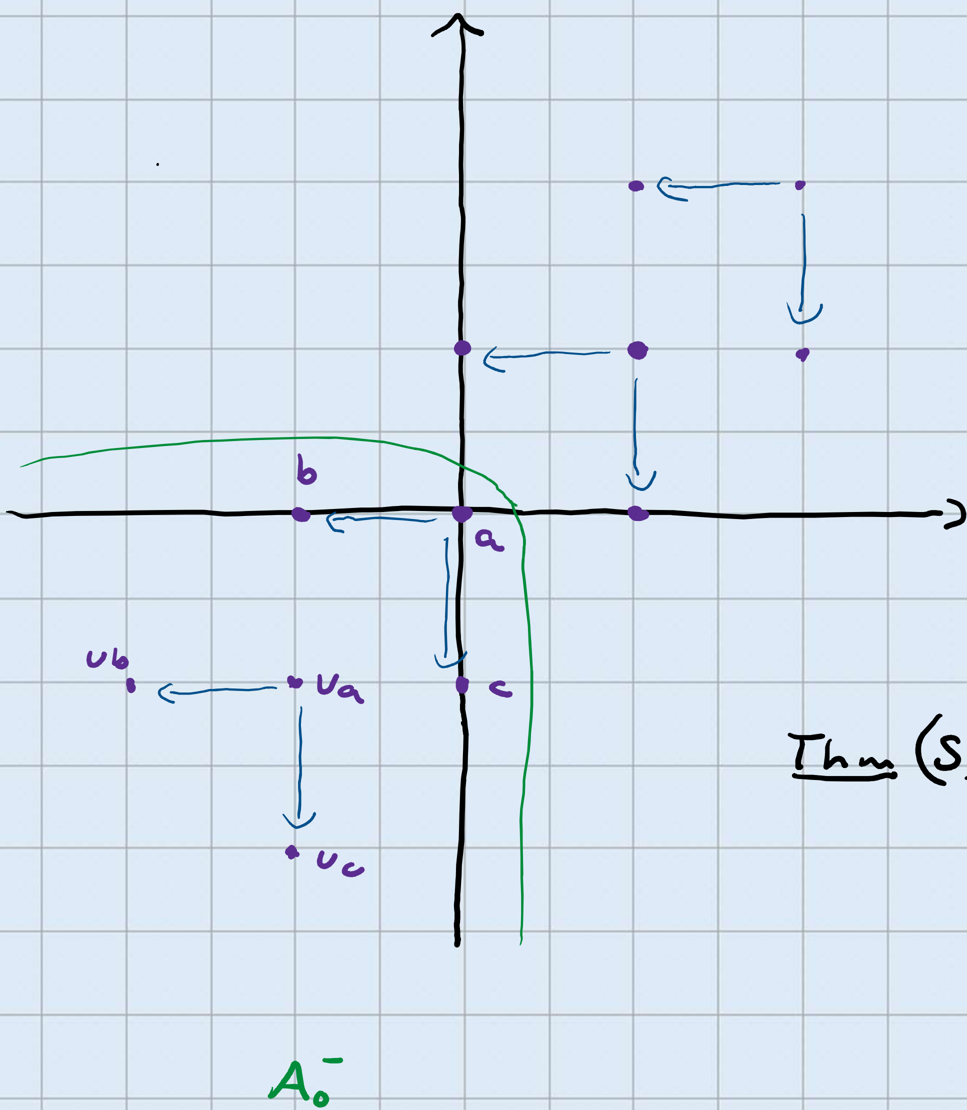
$$a_1a_2 + b_1a_2 \qquad b_1a_2 \qquad a_1b_2 \qquad b_1b_2 + a_1b_2$$

$$\begin{tikzcd}
c_1a_2 \arrow[d] & U(a_1c_2+c_1a_2) \arrow[d] \arrow[dr] & U(b_1c_2+c_1b_2) \arrow[d] \arrow[dl] & c_1b_2 \arrow[d] \\
U(a_1a_2+b_1a_2) & Ub_1a_2 & Ua_1b_2 & U(b_1b_2+a_1b_2)
\end{tikzcd}$$

$$\begin{tikzcd}
 & c_1c_2 \arrow[dl] \arrow[dr] & \\
U(a_1c_2+c_1a_2) & & U(b_1c_2+c_1b_2)
\end{tikzcd}$$

$$\begin{tikzcd}
c_1a_2 \arrow[d] & U(a_1c_2+c_1a_2) \arrow[d] \arrow[dr] & U(b_1c_2+c_1b_2) \arrow[d] \arrow[dl] & c_1b_2 \arrow[d] \\
U^2(a_1a_2+b_1a_2) & U^2b_1a_2 & U^2a_1b_2 & U^2(b_1b_2+a_1b_2)
\end{tikzcd}$$

$CFK^-(T_{2,3})$



- $CFK^\infty(K)$ free finitely gen'd $\mathbb{Z}\oplus\mathbb{Z}$-filtered cpx

- A_0^- = subcomplex in third quadrant

- $\iota_K : CFK^\infty(K) \longrightarrow CFK^\infty(K)$ skew-filtered graded chain map; $\iota_K^4 \simeq Id$

<u>Thm</u> $(S^3_{+1}(K), \iota) \underset{\text{e.e.}}{\simeq} (A_0^-, \iota_K)$