Math 4432 - Spring 2019
Homework 3

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 2, 3, 4, 5, 9.* **Due: February 18**

1. The product of Hausdorff spaces is Hausdorff.

2. Show that a space X is Hausdorff if and only if $\Delta = \{(x,x)|x \in X\}$ is closed in $X \times X$.

3. Let $X = [0,1], A = \{0,1\} \subset X$ and $Y = [4,5]$. Define the map $f : A \to Y$ by $f(0) = 4$ and $f(1) = 5$. Show that $X \cup_f Y$ is homeomorphic to S^1.

4. Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$. Show that the decomposition space of X defined as

   $$\mathcal{D} = \{S_r | r > 0\},$$

   where $S_r = \{(x,y)|x^2 + y^2 = r^2\}$, is homeomorphic to $\mathbb{R}$.

5. Let $X = S^1 \times [0,1]$ and consider the decomposition space

   $$\mathcal{D} = \{\{(x,t)\} : t \in (0,1] \text{ and } x \in S^1\} \cup \{(x,0) : x \in S^1\},$$

   that is the only non-trivial set in the decomposition is $S^1 \times \{0\}$. Prove that $\mathcal{D}$ is homeomorphic to D^2.
   Remark: You might also want to try to show that if S^1 is replaced by S^n then the analogous decomposition space is homeomorphic to D^{n+1}.

6. Let D^2 be the unit disk in $\mathbb{R}^2$ and S^2 be the unit sphere in $\mathbb{R}^3$. Show that the upper hemisphere of S^2 is homeomorphic to D^2 and similarly for the lower hemisphere. Use this to show that S^2 is homeomorphic to $D^2 \cup_g D^2$ for some homomorphism $g : S^1 \to S^1$ where $S^1 = \partial D^2$. Determine g.

7. Let D^2 be the unit disk in $\mathbb{R}^2$. Let $\mathcal{D}$ be a decomposition of $\mathbb{R}^2$ whose only non-trivial set is D^2. Show $\mathcal{D}$ is homeomorphic to $\mathbb{R}^2$.

8. Let D be a disk and I be an interval in ∂D. If Σ is a surface and $f : I \to \partial\Sigma$ is an embedding, then show the surface

   $$\Sigma \cup_f D$$

   is homeomorphic to Σ.
   Hint: It might be good to try to show that the space obtained from a disk and an annulus by gluing them along intervals in their boundary is homeomorphic to an annulus. You may assume, as discussed in class, that given any connected component B of $\partial\Sigma$ there is an open set U in Σ that contains B and is homeomorphic to $S^1 \times [0,1)$.

9. Show that for any connected surface Σ and points p and q in Σ there is a homeomorphism $h : \Sigma \to \Sigma$ such that $h(p) = q$.
   Hint: Fix p and consider the set

   $$S = \{q \in \Sigma \text{ such that there is a homeomorphism sending } p \text{ and } q\}.$$

Remark: Notice that from this problem you know for any non-empty surface there are an uncountable number if different homeomorphisms $\Sigma \to \Sigma$. Also, there are two extensions of this problem you might want to consider. 1) If M is a connected n-manifold and points p and q in M there is a homeomorphism $h : M \to M$ such that $h(p) = q$. 2) If M is a connected n-manifold, for $n > 1$, and $\{p_1, \dots .p_k\}$ and $\{q_1, \dots q_k\}$ are two collections of distinct points in M, then there is a homeomorphism $h : M \to M$ such that $h(p_i) = q_i$ for $i = 1, \dots, k$. What happens for $n = 1$? Is there any similar statement that can be made?