Math 4432 - Spring 2019
Homework 6 (Optional)

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 2, 3, 6, 7, 8.* **Due: April 17**

1. Given a continuous map $f : S^n \to X$, show that f is homotopic to a constant map if and only if f extends to a map $F : D^{n+1} \to X$.

2. Let D^2 be the unit disk in $\mathbb{R}^2$ and $S^1 = \partial D^2$. Show there does not exist a map $f : D^2 \to S^1$ that is the identity on S^1. (That is, that satisfies $f(x) = x$ for all $x \in S^1 \subset D^2$.)
   Hint: Think about the natural inclusion $g : S^1 \to D^2$.

3. Let X and Y be two path connected topological spaces. Write down a natural map $\psi : \pi_1(X) \oplus \pi_1(Y) \to \pi_1(X \times Y)$ and prove it is an isomorphism.

4. Compute $\pi_1(T^2)$ using problem 3, where T^2 is the torus. More generally, let $T^n = S^1 \times T^{n-1}$ (so, for example, $T^3 = S^1 \times S^1 \times S^1$). Compute $\pi_1(T^n)$.

5. We can regard $\pi_1(X, x_0)$ as base point preserving homotopy classes of maps of (S^1, pt) to (X, x_0). Let $[S^1, X]$ be the set of homotopy classes of maps S^1 to X (not necessarily base point preserving). There is a natural map
$$\Psi : \pi_1(X, x_0) \to [S^1, X]$$
   that just ignores the base points. Show that Ψ is onto if X is path connected. Also show that $\Psi([\gamma]) = \Psi([\lambda])$ if and only if there is some $g \in \pi_1(X, x_0)$ such that $[\gamma] = g^{-1}[\lambda]g$.

6. The infinite dihedral group D_∞ is given by the presentation $\langle x, y | y^2, xyxy \rangle$. Show that D_∞ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$.
   Hint: From class we know $\mathbb{Z}_2 * \mathbb{Z}_2$ has presentation $\langle a, b | a^2, b^2 \rangle$. Notice that x and ab both have infinite order.

7. Suppose that neither G or H is the trivial group. Show that $G * H$ is infinite and non-abelian.

8. Use the Seifert-Van Kampen Theorem to compute the fundamental group of the unit n-sphere S^n for $n > 1$.
   Hint: You might want to induct on n.

9. Recall we can think of the torus T^2 as the unit square in the first quadrant of $\mathbb{R}^2$ with opposite edges identified. Show that T^2 is also $\mathbb{R}^2$ modulo the equivalence relation:

$$(x, y) \sim (x', y') \iff (x, y) - (x', y') = (n, m),$$

   where n, m are any two integers. Another way to say this is that T^2 is $\mathbb{R}^2$ modulo the group of translations that preserve the integer lattice in $\mathbb{R}^2$. (This is just like thinking of S^1 as the interval with endpoints identified and as $\mathbb{R}^1$ modulo unit translation.)

10. Let A be a 2×2 matrix with integer entries

$$A = \begin{pmatrix} p & r \\ q & s \end{pmatrix}.$$

   This matrix takes the integer lattice in $\mathbb{R}^2$ into the integer lattice. Show this induces a continuous map

$$\phi_A : T^2 \to T^2.$$

   Compute $(\phi_A)_* : \pi_1(T^2) \to \pi_1(T^2)$. (By this I mean choose a basis for $\pi_1(T^2)$ and give a matrix expressing $(\phi_A)_*$.)

11. If $\det A = \pm 1$ then ϕ_A is a homeomorphism (you don't have to prove this unless you want to). Assume $\det A = -1$. Let M be the quotient space of $D^2 \times S^1 \cup D^2 \times S^1$ by the relation $(\theta, \psi) \in \partial(D^2 \times S^1)$ in the first $D^2 \times S^1$ is identified with $\phi_A(\theta, \psi)$ in the second $D^2 \times S^1$. This is a 3-manifold (you don't have to prove this unless you want to). Compute $\pi_1(M)$. Hint 1: Your answer should only depend on q. Hint 2: In the Seifert-Van Kampen Theorem take A to be one of the tori in the quotient space and B to be the other. The computation will be easier here, but you need to justify why it is OK to use these closed sets instead of open sets as the hypothesis of the theorem asks for.