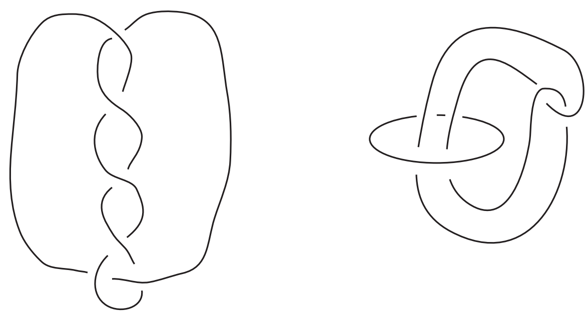
Math 4432 - Spring 2019
Homework 7

Work all these problems and talk to me if you have any questions on them.
Due: Never

1. Compute the fundamental group of the complement of the knot shown on the left of the figure. Show that it has a presentation with just two generators and one relation



2. Compute the fundamental group of the complement of the link on the right of the figure. (The formula to compute the fundamental group is just like the one for knots.)

3. What is the abelianization of the fundamental group from Problem 2. (Any guesses as to what the abelianization of the fundamental group of an n-component link might be? 2 Points Extra credit: Make a guess and prove it is correct.)

4. If G is the fundamental group of a knot complement then show that you can add one relation to a presentation of G that makes the presentation a presentation of the trivial group.

5. Let K be the figure eight knot. Is there a homomorphism from $\pi_1(X_K)$ onto the dihedral group D_3? Is there a homomorphism from $\pi_1(X_K)$ onto the dihedral group D_5? If so what is the homomorphism (this means write down a presentation for the group and then describe the homomorphism) and if not why?

6. Same question for $\pi_1(X_K)$ where K is the right handed trefoil knot.

7. Suppose that $p : \widetilde{X} \to X$ is a covering map. Show that the index $[p_*(\pi_1(\widetilde{X}, \widetilde{x_0})) : \pi_1(X, x_0)]$ equals the degree of the covering map.

8. Let X be a CW complex and let $\widetilde{X}$ be a degree n cover of X. Show that $\chi(\widetilde{X}) = n\chi(X)$. Here of course χ means Euler characteristic.
   Hint: Show each i-cell lifts to n different i-cells. The lifting criterion might be helpful.

9. If Σ_g is a genus g surface. For which g and h does Σ_g cover Σ_h?
   Hint: Use the previous problem to show when Σ_g does not cover Σ_h. The for the g and h where you can't prove a cover does not exist, try to explicitly construct a cover.

10. If F_n is the free group on n elements (that is has rank n), and G is a finite index subgroup of F_n of index m, then we know that G is a free group. What is its rank?
    Hint: recall F_n is the fundamental group of a wedge of n circles and subgroups correspond to covering spaces. Use Problem 8 to compute the Euler characteristic. Now how does the Euler characteristic relate to the fundamental group of a graph.
    Note: The rank of finite index subgroups of a free group is larger than the rank of a group. Is this true for infinite index subgroups? This is not part of the homework problem, just something to think about.

11. Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.
    Hint: If $f : X \to Y$ is a homeomorphism, then f restricted to $X - \{x\}$ is a homeomorphism from $X - \{x\}$ to $Y - \{f(x)\}$. Now think about the fundamental group.