Math 501 - Spring 2002
Homework 4

Work 7 of the 10 problems.

1. Let Σ be a regular, compact, orientable surface in $\mathbb{R}^3$ which is not homeomorphic to a sphere. Prove that there are points on Σ where the Gaussian curvature is positive, negative and zero.

2. Show that the equations of the geodesics in geodesic polar coordinates are given by

$$\rho'' - \frac{1}{2}(g_{22})_\rho(\theta')^2 = 0$$

$$\theta'' - \frac{(g_{22})_\rho}{g_{22}}\rho'\theta' + \frac{1}{2}\frac{(g_{22})_\theta}{g_{22}}(\theta')^2.$$

3. If p is a point of regular surface Σ prove that

$$K(p) = \lim_{r\to 0} \frac{12}{\pi}\frac{\pi r^2 - A}{r^4},$$

where $K(p)$ is the Gaussian curvature of Σ at p, r is the intrinsic radius of a circle $S_r(p)$ centered in p, and A is the area of the region bounded by $S_r(p)$.
HINT: Use Taylor expansion.

4. Show that in a system of normal coordinates centered at p all the Christoffel symbols are zero at p. (Recall normal coordinates are the image of Cartesian coordinates under the exponential map.)

5. Let $p \in \Sigma$ and $S_r(p)$ be the geodesic circle with center p and radius r. Let L be the length of $S_r(p)$. Prove that

$$L = 2\pi r - \frac{2\pi K(p)}{6}r^3 + R$$

where

$$\lim_{r\to 0}\frac{R}{r^3} = 0.$$

6. Let $p \in \Sigma$ and $S_r(p)$ be the geodesic circle with center p and radius r. Let L be the length of $S_r(p)$ and A the area of the region bounded by $S_r(p)$. Prove that

$$4\pi A - L^2 = \pi^2 K(p) r^4 + R$$

where

$$\lim_{r\to 0}\frac{R}{r^4} = 0.$$

Note this says that if $K(p) > 0$ the for small circles $4\pi A - L^2 > 0$ and similarly for $K(p) < 0$.

7. Prove the following formula for the Christoffel symbols in an arbitrary coordinate system u_1, u_2 (recall, the formulas in class were in an orthogonal coordinate system).

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial u_i} + \frac{\partial g_{il}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_l}\right),$$

where the matrix (g^{ij}) is the inverse of the matrix defining the metric (g_{ij}).

8. Determine the Christoffel symbols of a surface represented in the form $z = f(x, y)$. (You can use the formula in problem 7)

9. Denote by H, $\mathbb{R}^2$ with the metric from Homework 3 with Gauss curvature -1 and denote by E, $\mathbb{R}^2$ with its normal metric (curvature 0).

   a) Is it possible to have a rectangle in H with geodesic edges such that the sum of the interior angles is 2π?
   b) Is it possible to have an n sided polygon in E with geodesic edges such that the sum of the interior angles is 2π?
   c) Show that it is possible to have an n sided polygon in H with geodesic edges such that the sum of the interior angles is any preassigned, small number?
   d) Is it possible to have an n sided polygon in H with geodesic edges such that the sum of the interior angles is 2π?

10. In local coordinates write down the equations geodesics must satisfy in S^2 with the standard round metric (use stereographic coordinates). Do the same for H (as in problem 8).