Math 501 - Spring 2002
Homework 5

Work 6 of the 8 problems.

1. Let g be a Riemannian metric on Σ. A geodesic $\gamma : [0, \infty) \to \Sigma$ is called a ray leaving $\gamma(0)$ if
$$d(\gamma(0), \gamma(s)) = \text{length}(\gamma([0, s])),$$
for all $s > 0$. If Σ is not compact and g is complete show that for every $p \in \Sigma$ there is a ray leaving p.

2. Let $\Sigma = \{z = f(x, y) : (x, y) \in \mathbb{R}\}$ be a complete surface. Show that
$$\lim_{r\to\infty} (\inf_{x^2+y^2 \geq r} K(x, y)) \leq 0.$$

3. Derive a formula for the first variation of arc length without assuming the variation is proper.

4. If Σ is a complete surface, $p \in \Sigma$, $\gamma(s), s \in \mathbb{R}$ is a geodesic and $d(s)$ is the distance from $\gamma(s)$ to p then show that there is a point $s_0 \in \mathbb{R}$ such that $d(s_0) \leq d(s)$ for all s and that the geodesic joining p to $\gamma(s_0)$ is parpendicular to γ.

5. Let $\gamma : [0, l] \to \Sigma$ be a geodesic on a complete surface Σ and assume that $\gamma(l)$ is not conjugate to $\gamma(0)$. Let $w_0 \in T_{\gamma(0)}\Sigma$ and $w_1 \in T_{\gamma(l)}\Sigma$. Show there is a unique Jacobi field $J(s)$ along γ with $J(0) = w_0$ and $J(l) = w_l$.

6. Let $\gamma : [0, l] \to \Sigma$ be a geodesic parameterized by arc length and let $J(s)$ be a Jacobi field along γ with $J(0) = 0$ and $\langle J'(0), \gamma'(0)\rangle = 0$. Show that $\langle J(s), \gamma'(s)\rangle = 0$ for all $s \in [0, l]$.

7. Given the setup in problem 6. further assume that $|J'(0)| = 1$. Let $e_1(s)$ be the parallel transport of $\gamma'(0)$ along γ and $e_2(s)$ be the parallel transprot of $J'(0)$ along γ. They form an orthomormal basis for $T_{\gamma(s)}\Sigma$ and by 6. $J(s) = u(s)e_2(s)$ for some function $u(s)$. Show that the Jacobi equations for J can be written as
$$u''(s) + K(s)u(s) = 0,$$
with initial conditions $u(0) = 0$ and $u'(0) = 1$.

8. The unit sphere S^2 in $\mathbb{R}^3$ can be triangulated into 20 geodesic triangles of equal area. Moreover, all three angles in each one of these triangles are equal. How many triangles of this triangulation meet at each vertex?