Section V: Constructions

A. Branched Covers

a map $p: M \to N$ is a branched cover if $\exists$ a submanifold $B \subset N$ such that

$$p|_{(M - p^{-1}(B))} : (M - p^{-1}(B)) \to (N - B) \quad \text{is a covering map}$$

and

for each $x \in p^{-1}(B)$ there are coordinates about x and $p(x)$ in which p has the form

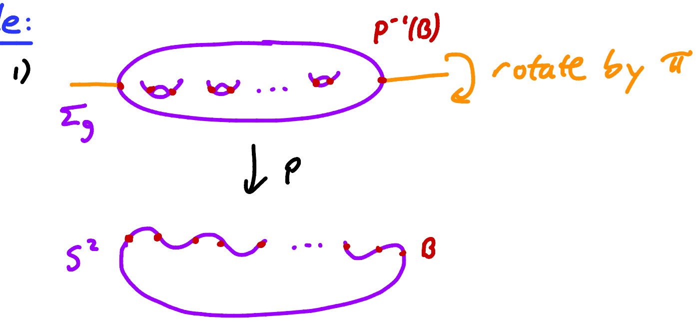
$$\mathbb{R}^{n-2} \times \mathbb{C} \to \mathbb{R}^{n-2} \times \mathbb{C}$$
$$(y, z) \mapsto (y, z^k) \qquad \text{some } k$$

and B and $p^{-1}(B)$ are $\mathbb{R}^{n-2} \times \{0\}$ in these coordinates

B is called the branch locus

example:

1)



2) Given any manifold X, then $p \times id : \Sigma_g \times X \to S^2 \times X$ is a branched cover. eg T^3 is a branched cover of $S^2 \times S^1$

lemma 1:

let $p: M \to N$ be a branched cover of compact manifolds with compact branch locus B

suppose 1) N has a symplectic form ω_N

2) B is a symplectic submanifold of (N, ω_N)

Then $\exists$ a symplectic form ω_M on M such that

$[\omega_M] = f^*[\omega_N] \in H^2(M; \mathbb{Z})$ and

$\omega_M = f^*\omega_N$ away from nbhd of $p^{-1}(B)$

Moreover, ω_M is well-defined upto isotopy

Proof: $f^*\omega_N$ is a closed 2-form and non-degenerate away from $p^{-1}(B)$

in particular, if $K_x = \ker dp_x \subset T_x M$ for $x \in B$ then K_x is an $\mathbb{R}^2$-bundle

and ω is only degenerate on K_x

if α is any exact 2-form that is non-degenerate and positive on K_x $\forall x \in p^{-1}(B)$

then for ε small enough $f^*\omega_N + \varepsilon\alpha$ is symplectic

to see such an α exists, take a coordinate chart about $x \in p^{-1}(B)$

and $p(x)$ such that p has the form

$$\mathbb{R}^{2n-2} \times \mathbb{C} \to \mathbb{R}^{2n-2} \times \mathbb{C}$$
$$(y, z) \longmapsto (y, z^k)$$

let $\alpha_x = d(\phi_x(y)\psi_x(z)\, u\,dv)$ where ϕ_x is a bump function on $\mathbb{R}^{2n-2}$ $=1$ near x

ψ_x " " $\mathbb{C}$ $=1$ near x

and $z = u + iv$

extend α_x by 0 off of support of $\phi_x\psi_x$

clearly α_x has desired property on the support of $\phi_x\psi_x$

now let α be the sum of α_x as x ranges over a finite set of $x \in p^{-1}(B)$

such that the supports of α_x cover $p^{-1}(B)$

clearly we have necissary α.

now ω_X is well-defined upto isotopy by Moser's Th$^{\text{m}}$ (Th$^{\text{m}}$ III.6)

since the space of α as above is convex

We can use the longuage of branched covers to extend Eliashberg's higher dimensional existence conjecture to 4D.

Conjecture:

If M a manifold of dimension 4 with

- a class $h \in H^2(M)$ s.t. $h \cup h \neq 0$ and
- a non-degenerate 2-form ω_0

Then $\exists$ an embedded surface $\Sigma \subset M$ such that

- $[\Sigma]$ is Poincaré dual to Nh for some large N
- there is some branched cover
$$p: \tilde{M} \to M$$
with branch locus Σ
- $\tilde{M}$ admits a symplectic structure ω s.t.
  1) $p^{-1}(\Sigma)$ symplectic
  2) $p^* h = [\omega]$
  3) ω is homotopic to $p^*\omega_0$ through 2-forms ω_t that are non-degenerate off of $p^{-1}(\Sigma)$ (maybe non-degenerate for $t \neq 1$)

Research Problem:

> Verify conjecture for $\#_{2n+1}\mathbb{C}P^2$ for $n>0$.
> Verify conjecture for other almost symplectic 4-manifolds we will discuss later

In dimension 4 we can weaken the notion of branched covers

call $p: M^4 \to N^4$ a branched cover if for each point $x \in M$ we have coordinate charts about x and $p(x)$ for which p has the form: $\mathbb{C}^2 \to \mathbb{C}^2$

$$(u,v) \longmapsto (u,v)$$
$$(u,v) \longmapsto (u^2,v) \qquad \text{simple branch point}$$
$$(u,v) \longmapsto (u^3-uv,v) \qquad \text{cusp}$$

exercise:

1) let $R = \{x \in M : dp_x \text{ not maximal rank}\}$

   (for points with 2^{nd} model map, this is just $\{u=0\}$
   for ones with 3^{rd} " " $\{v=3u^2\}$)

   show R is a smooth submanifold of M

2) $B = p(R)$ does not have to be a smooth submanifold of N
   it has "cusp points"
   near these points B is locally modeled on $\{27z_1^2 = 4z_2^3\} \subset \mathbb{C}^2$

3) for a generic p, B can also have isolated transverse double points

4) If N has a symplectic structure ω_N such that the coord. charts expressing p in model above pulls back the standard complex structure on $\mathbb{C}^2$ to one compatible with ω_N
   Then M has a canonical (upto isotopy) symplectic structure agreeing with $p^*\omega_N$ off of R.

   (M,ω_M) is called a symplectic branched cover over (N,ω_N)
   (need to distinguish from 1^{st} def^n by context)

Th^m (Auroux):

> any compact symplectic 4-manifold (M,ω) is a symplectic branched cover of $(\mathbb{C}P^2, \omega_{FS})$

Remark: We discuss a proof later.

Auroux has a way to construct invariants of (M,ω) from this Th^m
(maybe discuss later)

B Symplectic Cuts and blowups

let (M,ω) be a symplectic manifold

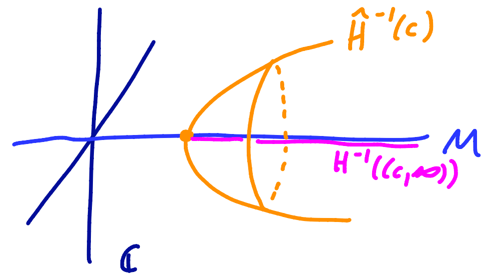
suppose we have a Hamiltonian circle action on (M,ω)

recall this means $\exists\, H: M \to \mathbb{R}$ st. the flow of its Hamiltonian vector field X_H (recall $\iota_{X_H}\omega = dH$) generates an S^1 action: $S^1 \times M \to M$

(recall flow tangent to level sets)

let c be a regular value of H and assume the S^1-action on $H^{-1}(c)$ is free

now consider $\hat{H}: M \times \mathbb{C} \to \mathbb{R} : (p,z) \mapsto H(p) - \frac{1}{2}|z|^2$



exercise:

1) $\hat{H}^{-1}(c) = [H^{-1}((c,\infty)) \times S^1] \cup H^{-1}(c)$

2) If we give $M \times \mathbb{C}$ the symplectic structure $\omega + dx \wedge dy$

then the Hamiltonian vector field of $\hat{H}$ is $X_H + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

Show the action is free and

$$\hat{H}^{-1}(c)/_{S^1} = H^{-1}((c,\infty)) \cup H^{-1}(c)/_{S^1}$$

so this is a smooth symplectic manifold (this is the same argument as we did for $\mathbb{C}P^n$)

and $H^{-1}(c)/_{S^1}$ and $H^{-1}((c,\infty))$ are symplectic submanifold

Hint: $\phi: H^{-1}((c,\infty)) \to \hat{H}^{-1}(c) \cap (M \times (0,\infty))$ is a symplectic embedding and transverse to S^1-action

$x \longmapsto (x, \sqrt{2(H(x)-c)})$

Remark: So we can put a smooth and symplectic structure on

$$H^{-1}([c,\infty))/_{S^1 \text{ action on } \partial}$$

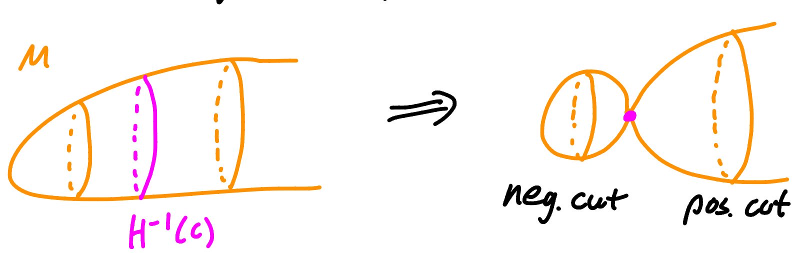
this is called the positive symplectic cut of M at c

Construction due to Lerman

We can similarly consider $\tilde{H}: M \times \mathbb{C} \to \mathbb{R} : (p,z) \mapsto H(p) + \frac{1}{2}|z|^2$

to get a symplectic structure on $H^{-1}((-\infty,c])/_{S^1 \text{ action on } \partial}$

this is the negative symplectic cut



exercise:

with notation as above note $H^{-1}(c)$ is a principal S^1-bundle (i.e. $U(1)$)

so $H^{-1}(c)$ has a Chern class $c_1(H^{-1}(c))$

$H^{-1}(c)/S^1$ is a co-dimension 2 submanifold of the positive cut

so its normal bundle ν_+ has structure group $U(1)$

Show $c_1(\nu_+) = c_1(H^{-1}(c))$

similarly $H^{-1}(c)/S^1$ has normal bundle ν_- in the negative cut

Show $c_1(\nu_-) = -c_1(H^{-1}(c))$

example:

Consider $M = \mathbb{C}^n$ and $H(x) = -\frac{1}{2}\|x\|^2$

let N be the negative symplectic cut at $c = -1/2$ (so $\|x\| = 1$)

i.e. $N = H^{-1}([0,1])/S^1$ action on $H^{-1}(1)$ $= B^{2n}/S^1$-action on ∂

note: $\partial B^{2n}/S^1 = \mathbb{C}P^{n-1}$ (by definition!)

exercise: $N = \mathbb{C}P^n$ (we see $\mathbb{C}P^n - \mathbb{C}P^{n-1} = \operatorname{int} B^{2n} \subset \mathbb{C}^n$ symplectically)

note: Cutting at other c still give $\mathbb{C}P^n$ but with scaled ω.

Now let N' be the positive symplectic cut

so $N' = \mathbb{C}^n - \operatorname{int} B^{2n}/S^1$ action on ∂

topologically we have a diffeomorphism $\mathbb{C}^n - \{0\} \longrightarrow \mathbb{C}^n - \{0\}$

$$z \longmapsto \frac{1}{\|z\|^2} z$$

this is orientation reversing so we see

$$N \cong -\mathbb{C}P^n - \{pt\}$$

and there is a symplectic structure on it!

let (M, ω) be any symplectic manifold

and $U \subset M$ be a Darboux chart symplectomorphic to an open ball in $\mathbb{C}^n$

we can perform the above construction on a ball in U to get a symplectic structure on $M \# (-\mathbb{C}P^n)$ this is called the

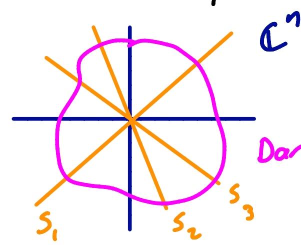
symplectic blowup of M

Remark:

1) a standard abuse of notation is to write this $M \# \overline{\mathbb{C}P}^n$ even though conjugating the complex structure does not always reverse the orientation

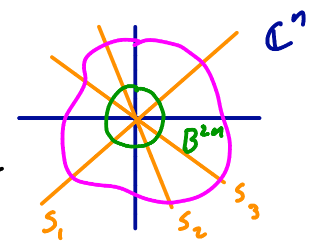
2) The smooth type of the blowup is well-defined, but the actual symplectic structure depends on the size of the ball used

note: Suppose we have a bunch of 2-dimensional symplectic submanifolds of M intersecting in a point p s.t. $\exists$ Darboux chart about p s.t. surfaces go to complex lines in $\mathbb{C}^n$



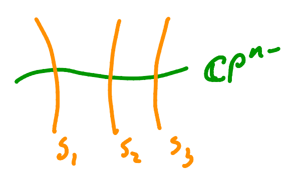
when we blow up we take a ball $B^{2n} \subset \mathbb{C}^n$

the submanifolds all $\pitchfork \partial B^{2n}$ in S^1 orbits so in the blow up these become disjoint points in $\mathbb{C}P^{n-1}$



i.e. S_i disjoint in $M \# \overline{\mathbb{C}P}^n$

(at least p removed form intersection)



Now given a symplectic manifold (M^{2n}, ω)

if there is an embedding of $(\mathbb{C}P^{n-1}, \lambda\,\omega_{FS})$ in (M^{2n}, ω) such that the normal bundle has Euler number -1

then $\mathbb{C}P^{n-1}$ has a neighborhood symplectomorphic to the $\mathbb{C}P^{n-1}$ in the positive symplectic cut above (Cor III.4)

thus $M^{2n} - \mathbb{C}P^{n-1}$ has a proper embedding of $\text{int}\, B^{2n}_{c+\varepsilon} - \text{int}\, B^{2n}_{c}$

so we can glue in $(\text{int}\, B^{2n}_{c+\varepsilon}, \omega_{std})$ to get a new symplectic manifold (M', ω') called the <u>symplectic blow down</u> of $\mathbb{C}P^{n-1}$ in M.

C. Normal Sums

Here is an interesting topological construction

let $f_1, f_2 : N^k \hookrightarrow M^n$ be 2 embeddings with disjoint images

let ν_i be the normal bundle of $f_i(N)$

this is an $\mathbb{R}^{n-k}$-bundle over N

suppose there is an orientation reversing bundle map

$\psi : \nu_1 \to \nu_2$ bundle isomorphism

so there are nbhds N_i of $f_i(N)$ diffeo to disk bundles in ν_i

exercise: Show ψ induces an orietation reversing diffeomorphism from ∂N_1 to ∂N_2 (still call it ψ)

let $\#_N M = M - (N_1 \cup N_2) /_{\sim \psi}$

this is called the normal sum of M along $f_1(N)$ and $f_2(N)$

if $M = M_1 \cup M_2$ and $f_i(N) \subset M_i$ then we write $M_1 \#_N M_2$

note: If $N = \{pt\}$, then $M_1 \#_N M_2 = M_1 \# M_2$ the ordinary connect sum!

so $M_1 \#_N M_2$ in general is like an N-parametric connect sum

exercise: If $f_i : N^{n-2} \hookrightarrow M^n$ are 2 embeddings

then ν_1 orientation reversing diffeomorphic to ν_2

$$\Leftrightarrow$$

$$c_1(\nu_1) = -c_1(\nu_2)$$

Th$^{\underline{m}}$ 2 (Gompf):

Suppose $f_i : (N, \omega_N) \hookrightarrow (M, \omega_M)$, $i = 1, 2$, are codimension 2 symplectic embeddings with disjoint images

Assume $c_1(\nu_1) = -c_1(\nu_2)$

For any choice of orientation reversing bundle isomorphism $\psi : \nu_1 \to \nu_2$ the normal sum $\#_N M$ has a symplectic structure ω such that outside of the gluing locus ω agrees with ω_M

exercise: Prove this if ν_2 trivial

Why does this only work for codimension 2?

Hint: S^n admits symplectic structure $\Leftrightarrow$ $n=2$

the proof is another application of symplectic cuts

let $\begin{array}{c} S^1 \to P \\ \pi \downarrow \\ M \end{array}$ be a principal S^1-bundle

elt of S^1

given $x \in P$ we get a map $\gamma_x : (-\varepsilon, \varepsilon) \to P : t \mapsto x \cdot e^{it}$

let $R(x) = \gamma_x'(0)$ this is a vector at x

and R is a vector field on P

note: flow of R is the S^1-action!

exercise:

1) $\exists$ a 1-form α on P such that $\alpha(R) = 1$

$\mathcal{L}_R \alpha = 0$

α called a connection 1-form

2) $\mathcal{L}_R d\alpha = 0$ and $\iota_R d\alpha = 0$ Hint: check 2nd formula on R and $\ker\alpha$

show this implies $\exists$ $\omega \in \Omega^2(M)$

such that $\pi^* \omega = d\alpha$

ω is called the curvature of α

only really need 1) but while we are at it

3) show $-[\omega/2\pi]$ is in $H^2(M;\mathbb{Z})$ and $e(P) = -[\omega/2\pi]$

simple case of Chern-Weil

Euler class

now suppose ω is a symplectic form on M^{2n}

and consider

$\mathbb{R} \times P$ and $\omega_P = \pi^* \omega + d(r\alpha) = \pi^*\omega + dr \wedge \alpha + r\, d\alpha$

r

note: $\omega_P^{\,n+1} = (n+1)\, dr \wedge \alpha \wedge (\pi^*\omega)^n + r(\text{other terms})$

let $v_1 \dots v_{2n}$ be linearly independent in $\ker\alpha$ at some pt x

clearly

$d\pi$ isomorphism $\ker\alpha$ to TM

$$(\pi^*\omega)^n(v_1 \dots v_{2n}) = \omega(\pi_* v_1 \dots \pi_* v_{2n}) \neq 0$$

so

$$dr \wedge \alpha \wedge (\pi^*\omega)^n\left(\tfrac{\partial}{\partial r}, R, v_1 \dots v_{2n}\right) \neq 0$$

$\therefore$ for r sufficiently small (say $r \in (-\varepsilon, \varepsilon)$)

$$(\omega_p)^{n+1}\left(\tfrac{\partial}{\partial r}, R, v_1 \dots v_{2n}\right) \neq 0$$

$\therefore$ ω_p non-degenerate!

i.e. a symplectic form on $(-\varepsilon, \varepsilon) \times P$

let $H : (0, \infty) \times P \to \mathbb{R}$ be projection

so $dH = dr$ and so the Hamiltonian vector field is $X_H = -R$

generates (minus) S^1-action

now $H^{-1}(0) = P$

so $H^{-1}(0)/_{S^1} \cong M$ and induced form is ω

and positive symplectic cut of $\mathbb{R} \times P$ at 0 is an $\mathbb{R}^2$-bundle over M

with a symplectic structure and "Euler class" $e(P)$

moreover, (M, ω) embeds as "zero section"

similarly negative cut of $\mathbb{R} \times P$ at 0 is an $\mathbb{R}^2$-bundle over M

with a symplectic structure and "Euler class" $-e(P)$

moreover, (M, ω) embeds as "zero section"

Proof of Th$^{\underline{m}}$ 2:

given the embeddings $f_i : (N, \omega_N) \to (M, \omega_M)$ from theorem

let ν_i be their normal bundles and P be unit S^1-bundle in ν_2

let $\omega_p = \pi^*\omega_N + d(r\alpha)$ on $(-\varepsilon, \varepsilon) \times P$ as above

let $C_{\pm 1} = \pm$ symplectic cut of $(-\varepsilon, \varepsilon) \times P$ at 0.

and $z_{\pm 1}$ the embedding of (N, ω_N) in $C_{\pm 1}$

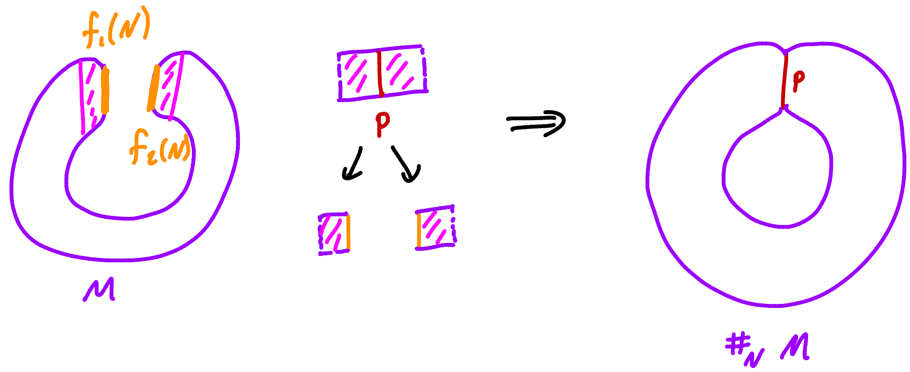
from Cor III.4 $f_1(N)$ has a neighborhood $U_{\pm 1}$ symplectomorphic to a nbhd of $Z_{(\pm1)^i}$ in $C_{(\pm1)^i}$

by shrinking ε can assume $C_{(\pm1)^i}$ is symplectomorphic to nbhd $U_{(\pm1)^i}$ of $f_1(N)$

note: $U_{(+1)} - f_2(N)$ symplectomorphic to $(0,\varepsilon)\times P$

$U_{(-1)} - f_1(N)$ " " $(-\varepsilon,0)\times P$

so we can glue $M - (f_1(N)\cup f_2(N))$ and $(-\varepsilon,\varepsilon)\times P$ to get a symplectic str on $\#_N M$



$\boxplus$

Cor 3 (Gompf):

> let G be any finitely presented group
> Then there is a closed symplectic $2n$-manifold $(n\geq 2)$ M with $\pi_1(M)\cong G$

Remark: There are strong restrictions on the fundamental groups of Kähler manifolds, eg we already saw b_1 must be even. So this result shows there are <u>many</u> more symplectic manifolds than Kähler.

Proof:

We construct a symplectic 4-manifold M then $M\times S^2\times\cdots\times S^2$ will realize G in any dimension $2n$, $n\geq 2$.

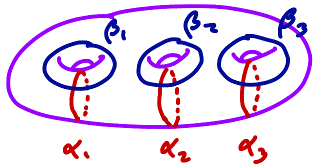
let $G = \langle g_1 \dots g_k \mid r_1 \dots r_\ell \rangle$

let Σ be a surface of genus k

and $\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_k$

be standard

generators of $H_1(\Sigma_k)$



note: $\pi_1(\Sigma_k) / \langle \beta_1 \dots \beta_k \rangle$ is the free group generated by $\alpha_1 \dots \alpha_k$

(connected to common base pt!)

let γ_i be an immersed curve in Σ_k realizing r_i in above quotient

(where α_j are substituted for g_j in r_i)

also let $\gamma_{\ell+i} = \beta_i$ for $i = 1 \dots k$

so $\pi_1(\Sigma_k) / \langle \gamma_1 \dots \gamma_{\ell+k} \rangle \cong G$

Claim: We can find a surface Σ and curves $\gamma_1 \dots \gamma_m$ such that

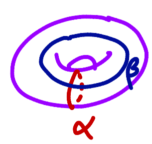
$\pi_1(\Sigma) / \langle \gamma_1 \dots \gamma_m \rangle \cong G$ and $\exists$ closed 1-form ρ on Σ

that restricts to a volume form on each γ_i

assuming this for now we build our manifold

start with $\Sigma \times T^2$ with a product symplectic form ω

let α, β be curves



in $T^2 = \alpha \times \beta$

let θ be angular coord on α

so $d\theta$ closed 1-form on T^2

note: $T_i = \gamma_i \times \alpha$ is Lagrangian $\forall i$

but if we set $\eta = \pi_1^* \rho \wedge \pi_2^* d\theta$ (where π_i are proj to 1^{st} and 2^{nd} factors)

then $\omega + t\eta$ is symplectic for small t

moreover T_i now symplectic!

note: We can perturb the T_i to be embedded and symplectic

indeed $\Sigma \times T^2 = (\Sigma \times \beta) \times \alpha$

and we can clearly C^∞-small perturb $\gamma_i \times \{\beta \cap \alpha\}$ in $\Sigma \times \beta$

to be embedded $\therefore$ when crossing these curves with

α we get a torus C^∞ close to original T_i (still call T_i)

since being symplectic an open condition the new T_i

still symplectic

exercise: normal bundles to all T_i trivial

Fact: in $E(1) = \mathbb{C}P^2 \#_9 \overline{\mathbb{C}P}^2$ $\exists$ an embedded symplectic torus T with trivial normal bundle and $\pi_1(E(1) - T) = \{1\}$

(maybe prove later but well-known fact)

now $\pi_1(\Sigma \times T^2) = \pi_1(\Sigma) \times \pi_1(T^2)$ gen by α, β

Van Kampen's Th$^{\underline{m}}$ says $\Sigma \times T^2 \#_{T_1 = T} E(1)$

has fundamental group $\pi_1(\Sigma) \oplus \mathbb{Z} / \langle \gamma_1 \rangle$ β

(note: $\pi_1(\partial(\Sigma \times T^2 - \text{nbhd}(T_1)) = \mathbb{Z}^3$ gen by α, γ_1, meridian)

so if we let $X = \Sigma \times T^2$ normal summed a copy of $E(1)$ for each T_i and an $E(1)$ for $\{x\} \times T^2$, we see $\pi_1(X) \cong G$ and X symplectic!

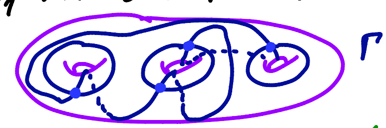
so we are done once we verify claim

exercise: think about base point

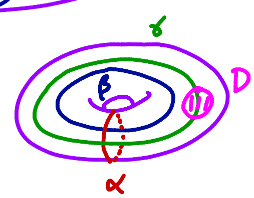
Proof of Claim:

for homological reasons our original $\gamma_1 \dots \gamma_{k+l}$ in Σ_k might not have such a β so we start by modifying our surface

we can assume γ_i intersect transversely so we get a graph $\Gamma \subset \Sigma_k$

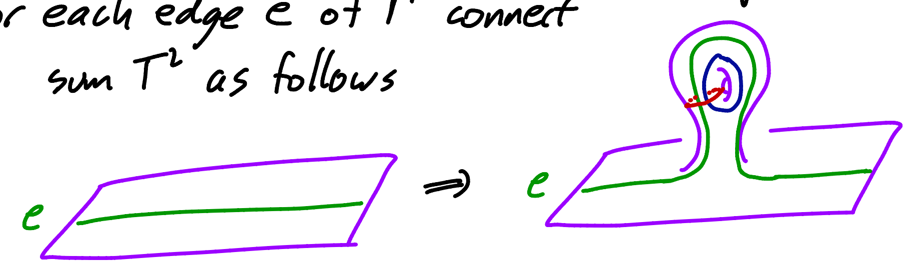


now consider $T^2 = \alpha \times \beta$



add a curve γ parallel to β and a disk D about pt on γ

for each edge e of Γ connect sum T^2 as follows



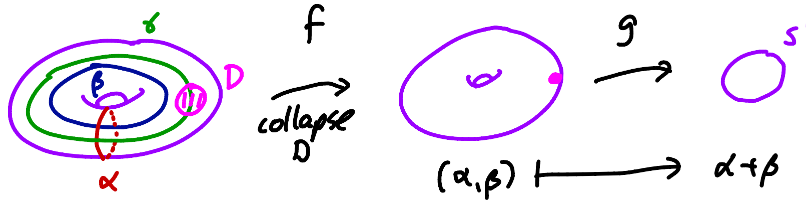
let Σ be $\Sigma_k \#$ all these tori and γ_i now be old γ_i ($\#$ with γ in T^2's) and the α and β from T^2's

clearly we still have $\pi_1(\Sigma) / \langle \gamma_i \rangle \cong G$

but also each edge in new graph Γ has a segment in a T^2
(on α, β or γ)

we now construct ρ

first note T^2 has a closed 1-form λ that has positive integral on α, β, γ and is 0 on D

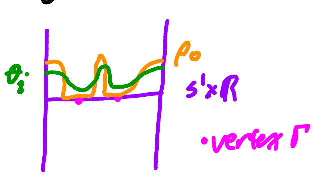


let $\lambda = (g \circ f)^* d\theta$

now let ρ_0 be the 1-form on Σ that equals λ on all T^2 and 0 elsewhere

note: $\int_e \rho_0 > 0 \quad \forall$ edges in Γ

1-forms on S^1 are $g\,d\theta$ so



thus for each γ_i, $\exists$ a volume form θ_i s.t. for each e in γ_i

$$\int_e \theta_i = \int_e \rho_0$$

so $\exists$ functions f_i on γ_i s.t. $df_i = \theta_i - \rho_0$ and f is 0 at each vertex of Γ

f_i define a function on Γ that extends to $F: \Sigma \to \mathbb{R}$

now $\rho = \rho_0 + dF$ is the desired form